NEW RESULTS ON DELAY-DEPENDENT STABILITY ANALYSIS AND STABILIZATION OF TIME-DELAY SYSTEMS

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by

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Under the guidance of

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This is to certify that the thesis entitled NEW RESULTS ON DELAY-DEPENDENT STABILITY ANALYSIS AND STABILIZATION OF TIME-DELAY SYSTEMS, submitted by DUSHMANTA KUMAR DAS to National Institute of Technology, Rourkela, is a record of bonafide research work under our supervision and we consider it worthy of consideration for award of the degree of Doctor of Philosophy of the Institute.

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DECLARATION

I certify that

a. The work contained in this thesis is original and has been done by me under the general supervision of my supervisors.

b. The work has not been submitted to any other Institute for any degree or diploma.

c. I have followed the guidelines provided by the Institute in writing the thesis.

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DUSHMANTA KUMAR DAS

Abstract

The interconnection between physical systems is accomplished by flow of information, energy and material, alternatively known as transport or propagation. As such flows may take a finite amount of time, the reaction of real world systems to exogenous or feedback control signals, from automatic control perspective, are not instantaneous. This results time-delays in systems connected by real-world physical media. Indeed, examples of time-delay systems span biology, ecology, economy, and of course, engineering. To this end, it is known that an arbitrary small delay may destabilize a stable system whereas, a delay in the controller may be used to stabilize a system that is otherwise not stabilizable by using a delay-free controller. In general, the presence of time-delay in a system makes the system dynamics infinite-dimensional, and analysis of such systems is complex.

This thesis investigates stability analysis and stabilization of time-delay systems. It proposes a delay-decomposition approach for stability analysis of systems with single delay that leads to a simple LMI condition using a Lyapunov-Krasovskii functional. Moreover, a static state feedback controller is designed for systems with state and input-delay using this delay-decomposition approach. Numerical comparison of the present results vis-à-vis the existing ones for the systems with constant delay considered shows that the present ones are superior. Next, a PI-type controller is implemented for systems with input-delay to improve the tolerable delay bound.

Other problems considered is to analyze the stability of systems with two delays. As the number of delays incorporated in the system dynamics increases, it becomes further complex for analysis. However, most of the approaches treated such problems by handling

the delay terms individually. A new approach is proposed to derive less conservative criteria for nominal and uncertain systems by exploiting the overlapping feature of the delays.

Finally, stabilizing ability of artificial delays incorporated in dynamic state feedback controller is investigated. A dynamic controller with state-delay is proposed to improve the tolerable delay bound of the system than that achievable using static and simple dynamic controller.

Key words: Time-delay systems, Lyapunov-Krasovskii functional, Discretization, Overlapping delay ranges, Static state feedback controller, PI-type state feedback controller, Artificial delay, CPPT.

Contents

List of symbols and acronyms

List of symbols

$\Re$	:	The set real numbers
$\Re^n$	:	The set of real n vectors
$\Re^{m\times n}$	:	The set of real $m \times n$ matrices
$\|\|X\|\|$	:	Euclidean norm of a vector or a matrix X
$\in$	:	Belongs to
$< (\leq)$	:	Less than (Less than equal to)
$> (\geq)$	:	Greater than (Greater than equal to)
$\neq$	:	Not equal to
$\forall$	:	For all
$\rightarrow$	:	Tends to
$y \in [a, b]$	:	$a \leq y \leq b$; $y, a, b \in \Re$
0	:	A null matrix with appropriate dimension
I	:	An identity matrix with appropriate dimension
X^T	:	Transpose of matrix X
X^{-1}	:	Inverse of X
$\lambda(X)$	:	Eigenvalue of X
$\lambda_{\max}(X)$	:	Maximum eigenvalue of X
$\lambda_{\min}(X)$	:	Minimum eigenvalue of X
λ_{real}	:	Real part of a eigenvalue
$\det(X)$	:	Determinant of X
$\mathrm{trace}(A)$	:	Trace of A

$\mathrm{diag}(x_1, \ldots, x_n)$	:	A diagonal matrix with diagonal elements as x_1,x_2,$\ldots$,x_n
$X > 0$	:	Positive definite matrix X
$X \geq 0$	:	Positive semidefinite matrix X
$X < 0$	:	Negative definite matrix X
$X \leq 0$	:	Negative semidefinite matrix X
$*$	:	The symmetric terms in a matrix is denoted by $*$
$\mathcal{C}[a, b]$	:	The set of $\Re^n$ valued continuous functions on [a, b]
$\mathcal{C}$	:	$\mathcal{C}[-r, 0]$
$\lvert\lvert\phi\rvert\rvert_c$	:	The continuous norm $\max\limits_{a \leq \xi \leq b} \lVert\phi(\xi)\rVert$ for $\phi \in \mathcal{C}[a, b]$.

List of acronyms

ARE	:	Algebraic Riccati Equation
ARI	:	Algebraic Riccati Inequality
LMI	:	Linear Matrix Inequality
LQR	:	Linear Quadratic Regulator
LTI	:	Linear Time-Invariant
LHS	:	Left Hand Side
RHS	:	Right Hand Side
CPPT	:	Continuous Pole Placement Technique
MTDB	:	Maximum Tolerable Delay Bound

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CHAPTER 1

Introduction

1.1 Background

A dynamical system, in general, is modelled as:

$$\dot{x}(t) = f\left(x(t), t\right), \quad t \in \Re^{+}, \tag{1.1}$$

where $x(t) \in \Re^n$ are known as the state variables. Let x_{eq} be an *equilibrium state* in the sense that $f(x_{eq}, t) = 0$, $\forall t \geq 0$ and the differential equation (1.1) characterizes the evolution of the state variables with respect to time. It is presumed that the future evolution of the system is completely determined by the current value of the state variables. Simply, the value of the state variables $x(t)$, $t_0 \leq t < \infty$, for any initial time t_0 can be found using the initial condition $x(t_0) = x_0$.

In reality, systems exist for which evolution of state variables $x(t)$ not only depends on the present values of $x(t)$, but also on their past values $x(\xi)$, $t - h \leq \xi \leq t$, $h \geq 0$. Such systems are called time-delay system [38, 43, 44, 98, 115, 132]. Time-delay systems are also called systems with after-effect or delay [136, 137, 139]. Broadly, the delay phenomenon appears in almost all the dynamics, e.g. biology, chemistry, population, economics, mechanics, physics, psychology, as well as in engineering. Some occurrences and effects of delay phenomenon in various systems are presented in Table 1.1.

For time-delay systems, evolution of the states is represented in a finite Euclidean space or

Table 1.1: Delay occurrences and effects in different processes

Processes	**Delay Occurrences**	**Effects**
Feedback Control [43]	During actuation, sensing, generating control signal	Performance degradation and instability.
Interconnected power systems [21]	Communication channel for sending area control error (control signal)	Oscillation and instability.
Network control system (NCS) [152]	Parallel computation and computer networking	Performance degradation and instability.
Supply-chain management system [140, 151]	Decision making, transportation-line delivery, manufacturing, etc.	Influence every stage of the supply-demand chain, deteriorate inventory regulation causing financial losses, inefficiencies, and reduces quality-of-service.
Milling processes [43]	At the interface of the metal work-piece and the cutting tool	Undesirable vibrations, known as regenerative chatter instability, leads to increased tool wear, undesirable surface quality, and reduces productivity.
Interconnected and distributed systems [116]	During sensing, actuation process and transmission of control signal	Performance degradation and maybe instability.
Tele-operation system [1]	During transmission of control signal	Instability.
Tele-surgery [154]	During transmission of control signal	Accuracy is very important, leads to death of the human being.
Breathing process [150]	Within the physiological circuit	Uncontrolled carbon dioxide level in the blood.
Population dynamics [83]	Maturity of offspring	Uncontrolled population growth.

in a functional space. The most widely used representation is by using functional differential equation [8, 43, 47, 114, 139]. A retarded functional differential equation takes the form

$$\dot{x}(t) = f(x_t, t), \quad t \in \Re^+, \tag{1.2}$$

where $x(t) \in \Re^n$ is the state; $x_t = x(t+\theta)$, $-h \leq \theta \leq 0$, $h > 0$ is the time-delay; $f(x_t, t)$: $\mathcal{C} \times \Re \to \Re^n$, where $\mathcal{C}$ is the set of continuous functions mapping from $\Re^n$ in the time-interval

$t-h \leq \phi \leq t$ to $\Re^n$. Clearly, if the evolution of $x(t)$ is sought at time instant $t \geq t_0$, then one must first know x_t for $-h \leq \theta \leq 0$, which therefore defines the initial condition and is denoted as $x_{t_0} \in \mathcal{C}$. The above representation is used in this work.

1.2 Classification of time-delay systems

In this thesis, works on linear time-delay systems are presented. Hence, herefrom, we consider only linear systems. According to commonly accepted denomination introduced by [8,74–76], time-delay system can be classified based on how the delay affects the evolution of the states, as the following.

1.2.1 Systems with discrete delays

For such systems, the state evolution depends on states at some specific past time-instants and can be represented as:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + A_h x(t-h_x) + Bu(t) + B_h u(t-h_u), \\
y(t) &= C_h x(t-h_y),
\end{aligned}$$

where $x(t)$ is the state, $u(t)$ is the input, $y(t)$ is the output, h_x is the state delay, h_u is the control input delay and h_y is the output delay.

An example of systems with discrete delays is a chemical process described as follows. The quantity of the product of an incomplete and non-instantaneous irreversible chemical reaction which produces a product P from the reactant R, can be increased by streaming process is an example of systems with discrete delays [8,117]. The whole process (i.e. reaction plus streaming) can be modelled by a system of nonlinear delay differential equations with discrete delays.

$$\begin{aligned}
\dot{R}(t) &= \frac{q}{v}\left[\lambda R_0 + (1-\lambda)R(t-h) - R(t)\right] - K_0 e^{-\frac{Q}{T}} R(t), \\
\dot{T}(t) &= \frac{1}{v}\left[\lambda T_0 + (1-\lambda)T(t-h) - T(t)\right]\frac{\Delta H}{C\rho} - K_0 e^{-\frac{Q}{T}} R(t) - \frac{1}{VC\rho} U\left(T(t) - T\omega\right),
\end{aligned}$$

where $R(t)$ is the concentration of the component R; $T(t)$ is the temperature; R_0, T_0 are initial values at $t=0$; $\lambda \in [0,1]$ is the recycle coefficient; $(1-\lambda)q$ is the recycle flow rate of the unreacted R; h is the transport delay and other terms are constants of the system.

Another example of such systems is in economics [8, 117], where the interaction between consumer memory and price fluctuation on commodity market can be described by a functional differential equation as

$$\ddot{x}(t) + \frac{1}{S}\dot{x}(t) + \dot{x}(t-h) + \frac{Q}{S}x(t) + \frac{1}{S}x(t-h) = 0,$$

where $x(t)$ denotes the relative variation of the market price of the commodity; Q, S are parameters of the model; h is the time that must elapse before a decision to alter production is translated into an actual supply.

Such models also arise in heat exchanger dynamics [15,181], traffic modelling [5,121], teleoperation systems [1, 149, 169], biology [26, 150, 163], network control systems [160, 168, 170], modelling of rivers [20,87], population dynamics [39,83], neural network [3,172], fuzzy system [113], any systems with delayed measurement [101, 151, 152], system controlled by delayed feedback [44, 152] etc.

1.2.2 Systems with distributed delay

Here, the delays act on state $x(t)$ or $u(t)$ in a distributed fashion as shown below.

$$\dot{x}(t) = Ax(t) + \int_{-h_x}^{0} A_h(\theta)x(t+\theta)d\theta + Bu(t) + \int_{-h_u}^{0} B_h(\theta)u(t+\theta)d\theta.$$

Distributed delay systems are systems where the delay does not have a local effect as in pointwise delay systems but acts in a distributed fashion over a delayed time interval. An example of such systems is the SIR-model (S = number susceptible, I = number infectious, and R = number recovered (immune)) [8, 57] in epidemiology, which is described as:

$$\begin{aligned} \dot{S}(t) &= -\beta S(t)I(t), \\ \dot{I}(t) &= \beta S(t)I(t) - \beta \int_{h}^{\infty} \gamma(\tau)S(t-\tau)I(t-\tau)d\tau, \\ \dot{R}(t) &= \beta \int_{h}^{\infty} \gamma(\tau)S(t-\tau)I(t-\tau)d\tau, \end{aligned}$$

The distributed delay is the time spent by infectious people before recovering from the disease and takes values over $[-h, +\infty]$. This delay may be different from person to person but obeys a probability density of $\gamma(h)$, which tends to 0 at infinity and integral over $[-h, +\infty]$ equal to 1.

1.2.3 Neutral delay systems

In neutral time-delay systems, the delay is present in the state derivative terms and is represented as :

$$\dot{x}(t) = f\left(x_t, t, \dot{x}_t, u_t\right), \tag{1.3}$$

or

$$F\dot{x}(t) = f\left(x_t, t, u_t\right), \tag{1.4}$$

where $F :\to \Re^n$ is a regular operator. The modeling of coupling between transmission lines and population dynamics is done using neutral delay systems.

The evolution of forests [8, 127] can be represented by neutral delay equation. The model is based on a refinement of the delay-free logistic (or Pearl-Verhulst equation)

$$\dot{x}(t) = rx(t)\left[1 - \frac{x(t-h) + c\dot{x}(t-h)}{K}\right],$$

where $x(t)$ is the population, r is the intrinsic growth rate and K is the environmental carrying capacity.

As the delay appears in the system dynamics, the system becomes infinite dimensional due to the infinite roots of its characteristics equation [43, 139]. Due to the presence of this delay, the control performance of the closed loop system degrades [43, 151]. Many times, it causes instability of the system [8, 152]. Therefore, the stability analysis of such systems is important for researchers. Next section briefly reviews the salient results available regarding stability analysis of linear systems with time-delay. The presence of this delay degrades the performance of the system.

1.3 Literature review on stability analysis of time-delay systems

Similar to developed theories for linear systems without time-delays, stability analysis of time-delay systems follows two approaches: frequency domain and time domain. Frequency domain approaches have been long in existence because of its simplicity and computational ease, which can be checked efficiently by plotting graphically a certain frequency-dependent measures. For example, frequency sweeping and matrix pencil tests give necessary and sufficient conditions for delay-dependent and delay-independent stability for systems with delays [43]. Compared with frequency-domain approaches, the time-domain approaches have some advantages: (i) non-linearities and time-varying uncertainties can more easily be handled, (ii) easier to extend for controller synthesis and filter design irrespective of number of inputs and outputs. In time-domain approach, the direct Lyapunov method is a powerful tool for stability analysis and stabilization of time-delay systems [43, 51]. The present work is based on the latter approach and, hence, we emphasize the same.

1.3.1 Stability definitions [37, 43]

Defining a state norm as $||x_t||_c = \max\limits_{t-h\leq\phi\leq t}||x(\phi)||$, the stability definitions for (1.2) in the sense of Lyapunov are as follows.

Definitions. 1. *The system (1.2) is* stable *if for a* $\epsilon > 0$ *there exists a* $\delta = \delta(t_0, \epsilon) > 0$ *such that* $||x_{t_0}||_c < \delta$ *implies* $||x_t||_c < \epsilon$ *for all* $t \geq t_0$.

2. *It is* uniformly stable *if for a* $\epsilon > 0$ *there exists a* $\delta = \delta(\epsilon) > 0$ *such that* $||x_{t_0}||_c < \delta$ *implies* $||x_t||_c < \epsilon$ *for all* $t \geq t_0$.

3. *It is* asymptotically stable *if there exists a* $\delta(t_0) > 0$ *such that* $||x_{t_0}||_c < \delta(t_0)$ *implies* $\lim\limits_{t\to\infty} x(t) = 0$.

4. *It is* uniformly asymptotically stable *if for every* $\epsilon > 0$ *there exists a* $\delta > 0$ *and a* $T(\epsilon) > 0$ *such that* $||x_t||_c < \epsilon$ *for all* $t \geq t_0 + T(\epsilon)$ *whenever* $||x_{t_0}||_c < \delta$.

5. *It is* bounded *if there exists a* $\beta > 0$ *such that* $||x_t||_c < \beta$, *where* β *may depend on each solution.*

6. *It is* uniformly bounded *if for any* $\alpha > 0$ *there exists a* $\beta = \beta(\alpha)$ *independent of* t_0 *such that if* $||x_{t_0}||_c < \alpha$, *then* $||x_t||_c < \beta$ *for all* $t \geq t_0$.

7. *It is* uniformly ultimately bounded *if there exists a* $\gamma > 0$ *and if corresponding to any* $\alpha > 0$ *there exists a* $T(\alpha) > 0$ *such that* $||x_{t_0}||_c < \alpha$ *implies* $||x_t||_c < \gamma$ *for all* $t \geq t_0 + T(\alpha)$.

1.3.2 Lyapunov stability theorems [37, 43]

Based on the stability definitions in §1.3.1, stability of (1.2) can be ascertained using the extensions of classical Lyapunov theorem. There are two different ways of interpreting the stability of the considered system: as an *evolution in a functional space* (Lyapunov-Krasovskii functionals) [80–82] or as an *evolution in the Euclidean space* (Lyapunov-Razumikhin functions) [138]. It is well known that LK approach yields less conservative results than LR approach [43, 46, 101]. Next, we describe LK approach and some stability criteria developed in literature based on this.

Lyapunov-Krsovskii theorem [37,43]. *The system (1.2) is uniformly stable if there exists a continuous differentiable function $V(x_t)$, $V(0) = 0$, such that*

$$z(||x(t)||) \leq V(x_t) \leq v(||x_t||_c), \tag{1.5}$$

and

$$\dot{V}(x_t) \leq -w(||x(t)||), \tag{1.6}$$

where z, v, w are continuous nondecreasing scalar functions with $z(0) = v(0) = w(0) = 0$ and $z(r) > 0, v(r) > 0, w(r) \geq 0$ for $r > 0$. If $w(r) > 0$ for $r > 0$, then it is uniformly asymptotically stable and if, in addition, $\lim_{r\to\infty} z(r) = \infty$, then it is globally uniformly asymptotically stable.

Let us consider a Linear Time-Invariant (LTI) system with time-delay as:

$$\dot{x}(t) = Ax(t) + A_h x(t-h), \tag{1.7}$$

where $x(t) \in \Re^n$ is the state and $h \in \Re^+$ represents the time-delay and satisfies $0 \leq h \leq \bar{h}$; $A \in \Re^{n\times n}$ and $A_h \in \Re^{n\times n}$ are matrices governing the influence of the instantaneous state and the delayed state respectively. Unlike the initial condition for ordinary differential equation, here, the system requires past state information over a time-segment as the initial condition, i.e., $\phi = x(t), t \in [\ -\bar{h} \quad 0\ ]$.

Suppose system (1.7) is nominally stable for $h = 0$, i.e., all the roots of the characteristic equation are on the left half of the complex plane. It is also well known that these roots are continuous in the delay argument h [43]. Then, one may gradually increase h from its zero value to obtain an upper bound of time-delay $\bar{h}$, up to which the system is stable [46]. Depending on the size of the delay ($\bar{h}$) (i.e finite or infinite), one may classify the system as:

- *delay-independently stable* if $\bar{h} \to \infty$,
- *delay-dependently stable* if $\bar{h}$ is finite.

It may be noted that

- For system (1.7) to be stable, it is necessary that $[A + A_h]$ must be Hurwitz,
- For system (1.7) to be delay-independently stable, it is further necessary that A also be Hurwitz.

Methods available for delay-independent and delay-dependent stability analysis of (1.7) are now presented. Depending on whether stability and stabilization criteria include information of time-delays, those are classified into two classes: delay-independent criteria and delay-dependent criteria. The later one is of more practical importance due to the fact that the delay present in a system is always finite and hence exhibits more physical significance. These criteria are usually less conservative than the former one, especially when the time-delay is small [68, 147].

1.3.3 Delay-independent stability analysis

This section presents the methods for delay-independent stability analysis of (1.7) using Lyapunov-Krasovskii theorem.

An energy functional for (1.7) may be chosen following [43] as:

$$V = x^T(t)Px(t) + \int_{t-h}^{t} x^T(\theta)Qx(\theta)d\theta, \quad P > 0, Q > 0. \tag{1.8}$$

Note that, the above is a Lyapunov-Krasovskii functional since it satisfies

$$\lambda_{\min}(P)||x(t)||^2 \leq V \leq \lambda_{\max}(P)||x(t)||^2 + \tau\lambda_{\max}(Q)||x_t||_c^2. \tag{1.9}$$

The $\dot{V}$ then becomes

$$\dot{V} = 2x^T(t)PAx(t) + 2x^T(t)PA_hx(t-h) + x^T(t)Qx(t) - x^T(t-h)Qx(t-h). \tag{1.10}$$

Now, to separate the $x(t)$ and $x(t-h)$ factors in the second term of the above, one may obtain

$$2x^T(t)PA_hx(t-h) \leq x^T(t)PA_hQ^{-1}A_h^TPx(t) + x^T(t-h)Qx(t-h). \tag{1.11}$$

Using (1.11) in (1.10) and then to satisfy (1.6) one obtains the resulting stability criterion as

$$PA + A^TP + Q + PA_hQ^{-1}A_h^TP < 0. \tag{1.12}$$

One may now check for existence of $P > 0$ and $Q > 0$ that satisfy (1.12) in order to ascertain stability of (1.7) [37, 43]. This may be carried out by converting (1.12) into an equivalent LMI.

The stability criterion (1.12) may also be suitably modified to design static state-feedback stabilizing controllers for LTI systems [37]. Stabilization criterion may also be derived in similar way if the system has delay in the input of the system [30, 31, 37]. Such delay-independent static state-feedback stabilization of systems with both state and input delays has been developed in [52, 69].

1.3.4 Delay-dependent stability analysis

If the available information on size of the delay can be utilized to obtain stability analysis results of time-delay system then it is called as delay-dependent stability analysis. It is obvious that delay-dependent stability analysis is less conservative than that of delay-independent ones [8, 15, 31, 51, 61, 77, 93, 96, 103, 164, 176].

Delay-dependent stability analysis based on Lyapunov-Krasovskii theorem appears to have first been used in [159] to estimate the tolerable delay bound for linear uncertain systems. An improved estimate of the same was obtained in [158] by optimizing the bounding inequalities used. Several developments have been made since then over the last two decades or so that we discuss next.

1.3.4.1 Using Lyapunov-Krasovskii theorem

A. Complete type LK functional and discretization approaches:

The necessary and sufficient conditions for stability of time-delay systems of type (1.7) are ascertained by using a complete type LK functional [29, 43].

$$\begin{aligned} V(\phi) = x^T(t)Px(t) + 2x^T(t)\int_{-h}^{0} Q(\xi)x(t+\xi)d\xi + \int_{-h}^{0} x^T(t+\xi)S(\xi)x(t+\xi)d\xi \\ + \int_{-h}^{0}\int_{-h}^{0} x^T(t+\xi)R(\xi,\eta)x(t+\eta)d\eta d\xi, \end{aligned} \tag{1.13}$$

where $P = P^T \in \Re^{n\times n}$, continuous differentiable matrix functions $Q(\xi) : [-h, 0] \to \Re^{n\times n}$, $R(\xi, \eta) = R^T(\xi, \eta) \quad with \quad R(\xi, \eta) : [-h, 0]^2 \to \Re^{n\times n}$, $S(\xi) = S^T(\xi) : [-h, 0] \to \Re^{n\times n}$.

However, numerical solution for such a functional is not computationally tractable. One way to take care of this is by discretizing the functional into a number of delay intervals (integral intervals) so that with increase in number of intervals the numerical solution approaches the analytical one [40,43,51]. The discretization approach proposed in [43] is based on dividing the domain of definition of matrix function Q, R and S into smaller region and the matrix functions are chosen to be continuous piecewise linear which reduces the choice of LK functional into choosing a finite number of parameters with larger number of parameters and improved approximation with larger number of intervals.

The delay interval h is divided into N segments $h_p = [\theta_p, \theta_{p-1}]$, $p = 1, 2, \ldots, N$ of equal length $\delta = {}^{h}\!/\!_{N}$. Then

$$\theta_p = -p\delta = -\frac{ph}{N}, \quad p = 1, 2, \ldots, N. \tag{1.14}$$

This also divides the square matrix function $S = [-h, 0] \times [-h, 0]$ into $N \times N$ small squares $S_{pq} = [\theta_p, \theta_{p-1}] \times [\theta_p, \theta_{p-1}]$. Each square is further divided into two triangles

$$\begin{aligned} T^u_{pq} &= \left\{ (\theta_p + \alpha\delta, \theta_q + \beta\delta) \,\middle|\, \begin{array}{l} 0 \le \beta \le 1, \\ 0 \le \alpha \le \beta \end{array} \right\}, \\ T^l_{pq} &= \left\{ (\theta_p + \alpha\delta, \theta_q + \beta\delta) \,\middle|\, \begin{array}{l} 0 \le \alpha \le 1, \\ 0 \le \beta \le \alpha \end{array} \right\}. \end{aligned}$$

The continuous matrix functions $Q(\xi)$ and $S(\xi)$ are chosen to be linear within each segment h_p, and the continuous function $R(\xi, \eta)$ is chosen to be linear within each triangular region T^u_{pq} or T^l_{pq}. Let $Q_p = Q(\theta_p)$, $S_p = S(\theta_p)$, $R_{pq} = R(\theta_p, \theta_q)$. These Functions are piecewise linear, they can be expressed in terms of their values at the dividing points using linear interpolation formula i.e., for $0 \le \alpha \le 1$, $p = 1, 2, \ldots, N$.

$$\begin{aligned} Q(\theta_p + \alpha h) &= Q^{(p)}(\alpha) = (1 - \alpha)Q_p + \alpha Q_{p-1}, \\ S(\theta_p + \alpha h) &= S^{(p)}(\alpha) = (1 - \alpha)S_p + \alpha S_{p-1}. \end{aligned}$$

And for $0 \le \alpha \le 1$, $0 \le \beta \le 1$, $p = 1, 2, \ldots, N$, $q = 1, 2, \ldots, N$.

$$R(\theta_p + \alpha\delta, \theta_q + \beta\delta) = R^{(pq)}(\alpha, \beta) = \begin{cases} (1 - \alpha)R_{pq} + \beta R_{p-1,q-1} + (\alpha - \beta)R_{p-1,q}, & \alpha \ge \beta. \\ (1 - \beta)R_{pq} + \alpha R_{p-1,q-1} + (\beta - \alpha)R_{p,q-1}, & \alpha < \beta. \end{cases}$$

Thus, the Lyapunov-Krasovskii functional is completely expressed with P, Q_p, S_p, R_{pq}, $p, q = 0, 1, \ldots, N$.

However, such methods are difficult to adopt for control and filtering problems [40,43,51].

Also, the number of decision variables increases almost exponentially with increase in number of intervals leading to computational burden for higher-order systems.

To resolve the complexity of integrated quadratic factors which are dependent on discretized values of the state in [43], an alternate delay decomposition technique is proposed in [40]. This can generate an infinite sequence of Lyapunov functionals and associated delay-dependent criterion which are dependent on the number of decomposition of the delay interval. As the number of decomposition grows, the derived criterion in [40] shows the conservatism reduction properties. A stability criterion is proposed in [40], by discretizing r times the interval h, $r \in I_+$, $h_0 = 0$, $h_i = \frac{ih}{r}$, where h is the delay of system (1.7), with the following property $h_r = h$, $h_{i+j} = h_i + h_j, \quad \forall\, (i,j) \in \{1, 2, \cdots, r\}$. The following stability criterion is based on the above delay decomposition technique proposed in [40].

Theorem 1.1. *[40] System (1.7) is stable for any h such that $0 \le h \le h_{mr}$ if there exist $P_r > 0$, $Q_{ri} > 0$, $R_{ri} > 0$, $\forall i \in \{1, 2, \cdots, r\} \in R^{rn \times rn}$ satisfying following LMI:*

$$B_r^{\perp T} \mu_r \left(h_m\right) B_r^{\perp} < 0, \tag{1.15}$$

where $B_r^{\perp T}$ is the orthogonal component of B_r,

$$B_r = \begin{bmatrix} 1 & -A_{d0} & -A_{d1} & -A_{d2} & \cdots & -A_{dr} & 0 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 1 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 \\ 0 & E_{r1} & -E_{r2} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & E_{r1} & -E_{r2} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & 0 & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & E_{r1} & -E_{r2} & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$E_{r1} = \begin{bmatrix} 0_{(r-1)n,n} & 1_{(r-1)n,n} \end{bmatrix}, E_{r2} = \begin{bmatrix} 1_{(r-1)n,n} & 0_{(r-1)n,n} \end{bmatrix},$$

$$\mu_r(h) = \begin{bmatrix} \sum_{i=1}^{r} h_i R_{ri} & P_r & 0 & 0 \\ P_r & \sum_{i=1}^{r} Q_{ri} & 0 & 0 \\ 0 & 0 & -Q_r & 0 \\ 0 & 0 & 0 & -R_r \end{bmatrix},$$

$$Q_r = diag\left(Q_{r1}, \cdots, Q_{rr}\right), R_r = diag\left(\frac{1}{h_1}R_{r1}, \cdots, \frac{1}{h_r}R_{rr}\right).$$

Remark 1.1. *The features of the above approach in [40] are: (i) it can be extended to robust stability analysis. (ii) the LK functional does not depend on the uncertain parameters and (iii) the developed criterion takes the advantage of parameter-dependent Lyapunov functionals. One of the major drawbacks of the same approach is that it can not be easily extended to stabilization problem.*

Another discrete delay-decomposition approach has been proposed in [51] by constructing a new simple quadratic LK functional which consist of two parts. First part of the functional is proposed in [49, 50] by avoiding the quadratic factor in the functional. The first part of the functional is given as follows:

$$V_{1^{st}Part}(x, x_t) = x^T(t)Px(t) + \int_{t-h}^{t} x^T(\xi)Qx(\xi)d\xi + \int_{t-h}^{t} (h - t + \xi)\,\dot{x}^T(\xi)\,(hR)\,\dot{x}(\xi)d\xi.$$

The second part of the functional in [51] holds the quadratic factors which has been claimed to fill the gap between the computational result and the analytical result. The second part of the functional is as follows:

$$V_{2^{nd}Part}(x, x_t) = \int_{t-\frac{h}{N}}^{t} z^T(\xi)Sz(\xi)d\xi + \int_{t-\frac{h}{N}}^{t} \left(\frac{h}{N} - t + \xi\right)\dot{x}^T(\xi)\left(\frac{h}{N}W\right)\dot{x}(\xi)d\xi$$

where $z^T(t) = \left[x^T(t) \quad x^T\left(t - \frac{h}{N}\right) \quad \cdots \quad x^T\left(t - \frac{(N-1)h}{N}\right)\right]$. Using such quadratic LK functional the following theorem is proposed in [51] for which computational result approaches the analytical one with increasing number of delay decomposition.

Theorem 1.2. *[51] The system (1.7) is stable for $h > 0$ and $N \geq 2$, if there exist real $n \times n$ matrices $P > 0$, $Q > 0$, $R > 0$, $W \geq 0$, and $S_{ii} = S_{ii}^T (i = 1, 2, \cdots, N)$, $S_{ij}(i > j; i =$*

$1, 2, \cdots, N-1; j = 2, \cdots, N)$, *such that*

$$S = S^T = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1N} \\ * & S_{22} & \cdots & S_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ * & * & * & S_{1N} \end{bmatrix},$$

and

$$\Xi = \begin{bmatrix} \Xi^{(1)} & \Xi^{(2)} & \Xi^{(3)} \\ * & -W & 0 \\ * & * & -R \end{bmatrix} < 0,$$

$$\Xi^{(1)} = \begin{bmatrix} \Xi^{(1)}_{11} & \Xi^{(1)}_{12} & S_{13} & \cdots & S_{1N} & PB + R \\ * & \Xi^{(1)}_{22} & \Xi^{(1)}_{23} & \cdots & S_{2N} - S_{1N-1} & -S_{1N} \\ * & * & \Xi^{(1)}_{33} & \cdots & S_{3N} - S_{2N-1} & -S_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & * & \Xi^{(1)}_{NN} & -S_{N-1N} \\ * & * & * & * & * & \Xi^{(1)}_{N+1N+1} \end{bmatrix},$$

$$\begin{aligned} \Xi^{(1)}_{11} &= A^T P + PA + Q - W - R + S_{11}, \Xi^{(1)}_{22} = S_{22} - S_{11} - W, \Xi^{(1)}_{33} = S_{33} - S_{22}, \\ \Xi^{(1)}_{NN} &= S_{NN} - S_{N-1N-1}, \Xi^{(1)}_{N+1N+1} = -S_{NN} - Q - R, \Xi^{(1)}_{12} = S_{12} + W, \Xi^{(1)}_{13} = S_{23} - S_{12}, \\ \Xi^{(2)} &= \begin{bmatrix} \frac{h}{N} W^T A & 0 & 0 & \cdots & 0 & \frac{h}{N} W^T B \end{bmatrix}^T, \Xi^{(3)} = \begin{bmatrix} hR^T A & 0 & 0 & \cdots & 0 & hR^T B \end{bmatrix}^T. \end{aligned}$$

The above discussed discretization technique of [43] and the delay-decomposition techniques of [40,51] have two major drawbacks. First, the number of decision variables increases with increase in number discretizations. This increases the computational complexity of the stability criterion. The next major drawback is that the filter and controller design are difficult using these decomposition techniques.

On the other hand, simple LK functionals are used to obtain sufficient conditions. The motivation for using such functionals are: (i) these are easily extendable to control and filtering problems, and (ii) they reduces the computational burden invariably. A body of research publications have been made on reducing conservativeness of such analysis in the past decade [33, 53, 56, 111, 145, 146]. In all these attempts, progressively less conservative stability criterion have been obtained by suitably approximating either an integral term and/or the factors involving delay term in the derivative of LK functional. It is shown in [86] that several of such stability criterion are in fact equivalent and later in [8, 9] that several of the integral inequalities used in such approaches are also equivalent, but suitable one based on their affineness on the delay term may be used to obtain less conservative convex stability criterion. Next, we discuss some available approaches based on simple LK functional.

B. Simple type LK functional approaches:

It is a challenging issue to obtain a less conservative result by using simple type LK functional by avoiding integral terms in the derivative of the energy functional. A very commonly used simple LK functional is presented as follows:

$$V = x^T(t)Px(t) + \int_{t-h}^{t} x^T(\theta)Qx(\theta)d\theta + \int_{t-h}^{t}\int_{t+\theta}^{t} \dot{x}^T(s)R\dot{x}(s)dsd\theta. \tag{1.16}$$

Then, computing the derivative of the functional (1.16),

$$\dot{V} = 2x^T(t)P\dot{x}(t)+x^T(t)Qx(t)-x^T(t-h)Qx(t-h)+h\dot{x}^T(t)R\dot{x}(t)-\int_{t-h}^{t} \dot{x}^T(t)R\dot{x}(t)dt. \tag{1.17}$$

A huge literature are available [28, 43, 70, 71, 112, 134, 135, 146, 147] on stability analysis and stabilization results using simple type LK functional. Some of important approaches are discussed in the following.

- **Model-transformation approach** Model-transformation approaches have been introduced early in the stability analysis of time-delay systems. They transform a time-delay system into a new system, which is referred to as a comparatively similar system. Finally, the stability of the original system is determined through the stability analysis of the transformed model. The transformed model may be of different types, (uncertain) finite dimensional linear systems [8,43,70,71,177], time-delay systems [8,27,32,43].

  Model-transformation approach type 1 has been used by [43, 56, 72, 76, 117, 158, 159] For the purpose, the Leibnitz-Newton formula has been used to provide the difference between the instantaneous state and the delayed state as

$$x(t) - x(t-h) = \int_{t-h}^{t} \dot{x}(\theta)d\theta. \tag{1.18}$$

Using the above, one may replace the $x(t-h)$ term in (1.7) to obtain

$$\dot{x}(t) = (A + A_h)x(t) - A_h \int_{t-h}^{t} \dot{x}(\theta)d\theta. \tag{1.19}$$

The above clearly reflects the necessary condition for delay-dependent stability that $[A + A_h]$ is Hurwitz. Further, replacing $\dot{x}(\theta)$ in the last term of (1.19) using (1.7), one obtains

$$\dot{x}(t) = (A + A_h)x(t) - A_h \int_{t-h}^{t} (Ax(\theta) + A_h x(\theta - h))d\theta. \tag{1.20}$$

Clearly, the initial condition corresponding to (1.20) must now be $\phi_1 = x(\theta), \theta \in [-2\bar{\tau}, 0]$. Note that, the initial condition involved in system (1.7) is a subset of the one in (1.20). Therefore, ensuring stability of (1.20) ensures that of (1.7) but the reverse is not true [44]. The transformation from (1.7) to (1.20) is called a *first order transformation* and, in a similar way, higher order transformations may also be obtained [45, 73].

The model transformation approach is conservative as it may introduce additional poles which are not present in the original system, and one of these additional poles may cross the imaginary axis before any of the poles of the original system do as the delay increases from zero [44, 48].

Model-transformation approach type 2 This model transformation [75, 120] improves the result obtained from the Leibniz-Newton formula by introducing a free parameter C to be chosen adequately:

$$Cx(t-h) = Cx(t) - C \int_{t-h}^{t} \dot{x}(\theta)d\theta. \tag{1.21}$$

$$C\dot{x}(t) = (A + C)\, x(t) + (A_h - C)\, x(t-h) - C \int_{t-h}^{t} [Ax(s) + A_h x(s-h)]ds. \tag{1.22}$$

For $C = 0$: Original System is recovered. For $C = A_h$: the system obtained from

Leibnitz Newton formula is recovered as

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ A + A_h & -I \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \int_{t-h}^{t} \begin{bmatrix} 0 & 0 \\ 0 & -A_h \end{bmatrix} \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} ds. \tag{1.23}$$

Descriptor model transformation This model transformation has been introduced in [27, 32]. It does not introduce any additional dynamics.

$$\varepsilon \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + A_h \int_{t-h}^{t} \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} ds, \tag{1.24}$$

where

$$\varepsilon = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & I \\ A + A_h & -I \end{bmatrix}, A_h = \begin{bmatrix} 0 & 0 \\ 0 & -A_h \end{bmatrix}.$$

This approach is based on a bounding technique of cross terms involving a positive matrix. Involving Parks bounding technique leads to less conservative stability conditions coupled with complete LK functional [29]. Although this method is interesting and leads to quality results, it still leads to cross terms which are difficult to bound and result in conservative conditions.

- **Park's bounding approach** A more accurate bounding of cross terms in the derivative of the LK functional has been introduced in [124, 125]. The following lemma is the stability criterion using Park's bounding approach.

Lemma 1.1. *[124, 125] Assume that $a(\alpha) \in \Re^{n_x}$ and $b(\alpha) \in \Re^{n_y}$ are given for $\alpha \in \Omega$. Then, for any positive definite matrix $X \in \Re^{n_x \times n_x}$ and any matrix $M \in \Re^{n_y \times n_y}$, the following holds*

$$-2 \int_{\Omega} b^T(\alpha) a(\alpha) d\alpha \leq \int_{\Omega} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix}^T \Psi \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix} d\alpha,$$

where $\Psi = \begin{bmatrix} X & XM \\ M^T X & (M^T X + I) X^{-1} (XM + I) \end{bmatrix}$.

The above lemma is able to provide a tighter bound on the cross term which improves conservatism.

- **Jensen's inequality approach** Jensen's inequality has been used in [40, 49] to avoid the cross terms. The following lemma is the extension of the Jensen's Inequality

Lemma 1.2. *[179] For any constant matrix $0 < R$, $0 \leq \alpha < \beta$ and $0 < \gamma = \beta - \alpha$ the following bounding inequality holds:*

$$-\int_{t-\beta}^{t-\alpha} \dot{x}^T(\theta)R\dot{x}(\theta)d\theta \leq \gamma^{-1}\begin{bmatrix} x(t-\alpha) \\ x(t-\beta) \end{bmatrix}^T \begin{bmatrix} -R & R \\ * & -R \end{bmatrix} \begin{bmatrix} x(t-\alpha) \\ x(t-\beta) \end{bmatrix}. \tag{1.25}$$

Note that, RHS of the above is nonconvex in γ and may require approximation while deriving a convex criterion involving uncertain γ. An equivalent representation that may be used with benefit for such cases is using free variable matrices and expressed as:

$$\begin{aligned} &-\int_{t-\beta}^{t-\alpha} \dot{x}^T(\theta)R\dot{x}(\theta) \\ \leq &\begin{bmatrix} x(t-\alpha) \\ x(t-\beta) \end{bmatrix}^T \left\{ \begin{bmatrix} M + M^T & -M + N^T \\ * & -N - N^T \end{bmatrix} + \gamma \begin{bmatrix} M \\ N \end{bmatrix} R^{-1} \begin{bmatrix} M \\ N \end{bmatrix}^T \right\} \begin{bmatrix} x(t-\alpha) \\ x(t-\beta) \end{bmatrix}, \end{aligned} \tag{1.26}$$

where M and N are free weighting matrices of appropriate dimensions. Note that, with the choice $M = M^T = -N = -N^T = -\gamma^{-1}R$ in (1.26), one obtains (1.25).

The following Theorem presents a delay-dependent stability condition in the line of the result in [145] for (1.7) using Lemma 1.2.

Theorem 1.3. *[145] System (1.7) is asymptotically stable if there exists matrices, $P > 0$, $Q > 0$ and $R > 0$, satisfying the following LMI condition:*

$$\begin{bmatrix} PA + A^T P + Q + hA^T RA & PA_h + hA^T RA_h & 0 \\ * & -Q + hA_h^T RA_h & 0 \\ * & * & h^{-1}R \end{bmatrix} < 0, \tag{1.27}$$

Proof. Let us consider a suitable L-K functional as (1.16). Then, computing the derivative of the functional (1.16) as (1.17). To get a tighter solution, the bound for the integral term i.e.

$$-\int_{t-h}^{t} \dot{x}^T(t)R\dot{x}(t)dt \leq -h^{-1}\left(\int_{t-h}^{t} \dot{x}(s)ds\right)^T R\left(\int_{t-h}^{t} \dot{x}(s)ds\right)$$

using inequality (1.25) can be used to replace the integral term from (1.17) and replacing $\dot{x}(t)$ by system dynamics. Then, above condition is derived by considering the states as $\begin{bmatrix} x^T(t) & x^T(t-h) & \int_{t-h}^{t} \dot{x}^T(s)ds \end{bmatrix}$. □

The inequality (1.25) of Lemma 1.1 can be directly used to derive the following stability criterion.

Theorem 1.4. *[145] System (1.7) is asymptotically stable if there exists matrices, $P > 0$, $Q > 0$ and $R > 0$, satisfying the following LMI condition:*

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} \\ * & \Theta_{22} \end{bmatrix} < 0, \tag{1.28}$$

where

$$\begin{aligned} \Theta_{11} &= PA + A^TP - h^{-1}R + Q + hA^TRA, \Theta_{12} = PA_h - h^{-1}R + hA^TRA_h, \\ \Theta_{22} &= -Q - h^{-1}R + hA_h^TRA_h. \end{aligned}$$

Proof. Let us consider the same L-K functional as (1.16). Then, computing the derivative of the functional. To get a tighter solution, the Jensen's inequality (1.25) can be used to replace the integral term from the derivative of the functional and replacing $\dot{x}(t)$ by system dynamics. Then, condition (1.28) is derived by considering the states as $\begin{bmatrix} x^T(t) & x^T(t-h) \end{bmatrix}$. □

To obtain less conservative criterion, matrix variables are involved to obtain an equivalent representation of Jensen's inequality as (1.26) of Lemma 1.2. The following stability condition is derived using (1.26) of Lemma 1.2.

Theorem 1.5. *[145] System (1.7) is asymptotically stable if there exists matrices, $P > 0$, $Q > 0$, $R > 0$ and arbitrary matrices M, N satisfying the following LMI* *condition:*

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} & 0 \\ * & \Theta_{22} & 0 \\ * & * & -h^{-1}R \end{bmatrix} < 0, \tag{1.29}$$

where

$$\begin{aligned} \Theta_{11} &= PA + A^T P - h^{-1}R + Q + hA^T RA + (M + M^T), \\ \Theta_{12} &= PA_h + hA^T RA_h + (-M + N^T), \Theta_{22} = -Q + hA_h^T RA_h + (-N - N^T). \end{aligned}$$

Proof. The proof is similar to the proof of Theorem 1.4. One can use (1.26) of Lemma 1.2 to obtain the criterion (1.29). □

- **Free weighted matrices approach** In this approach, some weighted matrices are introduced in order to add some degree of freedom as result of which the additional constraints are included in the LMI [56]. These matrices are included by some zero equality expressions governed by the system dynamics. The following conditions hold for any G_i and T_i , $i = 1, 2, 3$ corresponding to (1.7).

$$2\left[x^T(t)G_1 + x^T(t-h)G_2 + \dot{x}^T(t)G_3\right]\left[x(t) - x(t-h) - \int_{t-h}^{t} \dot{x}^T(\theta)d\theta\right] = 0, \tag{1.30}$$

$$2\left[x^T(t)T_1 + x^T(t-h)T_2 + \dot{x}^T(t)T_3\right]\left[\dot{x}(t) - Ax(t) - A_h x(t-h)\right] = 0, \tag{1.31}$$

The following theorem is the delay-dependent condition using the above zero equality expression and Jensen's inequality.

Theorem 1.6. *[8] System (1.7) is asymptotically stable if there exists matrices $P > 0$, $Q > 0$, $R > 0$ and arbitrary matrices G_i, $i = 1, 2.$, satisfying the following LMI condition:*

$$\begin{bmatrix} \Upsilon & PA_h + hA^T RA_h - G_1 + G_2^T & -G_1 + G_3^T \\ * & -Q + hA_h^T RA_h - G_2 - G_2^T & -G_2 - G_3^T \\ * & * & h^{-1}R - G_3 - G_3^T \end{bmatrix} < 0, \tag{1.32}$$

where $\Upsilon = PA + A^T P + Q + hA^T RA + G_1 + G_1^T$.

Proof. Let us consider a suitable L-K functional as (1.16). Then, computing the derivative of the functional which is same as (1.17). To get a tighter solution, the Jensen's inequality (1.25) can be used to replace the integral term and zero equality constraint (1.30) is added to the derivative term to get tighter result. Then, the above condition (1.32) is derived by considering the states as $\begin{bmatrix} x^T(t) & x^T(t-h) & \int\limits_{t-h}^{t} \dot{x}^T(s)ds \end{bmatrix}$. □

- **Stability analysis of systems with interval-delay** As delay appears in ranges, it will vary from a non-zero lower bound to a upper bound, i.e. $h_1 \leq h \leq h_2$. Usually, the lower bound of the delay is considered to be zero in many literature [27,29,31–33], but in some special cases such as networked control systems which are basically feedback control systems with feedback loop closed through real-time communication channels, considers non-zero lower bound [131, 133]. Such systems are referred as an interval-delay systems. To investigate the delay-dependent stability of systems with interval-delay, the comparison theorem and matrix measure are employed in [98]. On the front of using LK functional based approaches, a stability result for systems with interval-delay is proposed in [62] by introducing some relaxation matrices in the derivative of the LK functional. By proposing an appropriate LK-functional without ignoring some useful terms stability result is obtained in [54]. In [171], Newton-Leibniz formula is used to obtain delay-independent and delay-dependent stability criteria for systems with interval-delay. However, in this analysis, some useful terms are neglected while dealing with the time-derivative of the LK functional which leads to a conservative result. By introducing free-weighted matrices and bounding technique in the delay range-dependent stability criterion, a stability result is derived in [84]. To obtain a less conservative result, further modifications in the choices of LK functional is considered in [131,133] which takes delay-range information into account appropriately, and a tighter bounding condition is used in time-derivative of the functional. By implementing a tighter bound in the derivative of the LK functional and weighted matrix variable approach, a less conservative result is obtained in [147].

The above discussion gives a brief idea about the stability analysis of time-delay systems. Another challenging objective of this thesis is to design less conservative stabilization controller for time-delay systems. Next section presents a discussion on existing stabilization approaches for time-delay systems.

1.4 Literature review on stabilization of time-delay systems

Stabilization problem is referred to as designing controller while ensuring stability of the system. As it is discussed earlier about the challenges on stability issue involved in time-delay system, control design for such systems to stabilize becomes tedious job for researchers. Most of the literature on stabilization of time-delay systems are mere extension of stability analysis approaches. Here also, the stabilization results are broadly classified as delay-independent and delay-dependent. The delay-independent stabilization provides a controller which can stabilize a system irrespective of the size of the delay. On the other hand, delay-dependent stabilization is concerned with size of the delay and usually provides an upper bound of the delay such that the closed loop system is stable for any delay less than the upper bound. Stabilization problem for time-delay systems can be classified as (i) stabilization of systems with state delay, and (ii) stabilization of systems with input delay, according to the association of delay term in the state or input of the system.

1.4.1 Stabilization of systems with state delay

A state delay system can be represented as

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + B_2 u(t), \tag{1.33}$$

where h is the delay associated with the state of the system, $u(t)$ is the control input to the system.

In studies of stabilization problem of state delay system, Riccati equation and Lyapunov approaches adopted in [16, 60, 148] to obtain delay-dependent results. An LMI approach is employed to tackle stabilization problem of time-delay systems in [11, 12, 88]. The LMI approach has two advantages. First, it needs no tuning of parameters and/or matrix. Second, it can be efficiently solved numerically using interior-point algorithms. For developing the stabilization criterion for time-delay systems, the model transformation approach is used in [44, 67]. In this approach, a delay-dependent criterion is developed for the delay free transformed model of the time-delay system. As the transformed model is not equivalent model of the time-delay system, this delay-dependent stabilization condition is conservative. Another reason for conservative result is the bounding method used to derive the bounds on weighted cross products of the state and its delayed version while trying to secure a negative value to the derivative of the corresponding LK functional. To take care both these issues, a descriptor model transformation technique proposed in [27,31,33] and the bounding technique proposed in [111] are combinedly used in [34]. An integral inequality approach is proposed in [179] to obtain delay-dependent stabilization criteria for systems with state delays. The state transformation approach is introduced to describe the delay-dependence dynamics and some control design scheme based on H_2 performance, H_∞ and simultaneous H_2/H_∞ criteria are established in [100,103]. A new state transformation approach is reported in [100] to facilitate the control design problem and reduce the conservatism. To reduce the conservatism numerous approaches such as free-weighted matrix variable approach [53,92,95], introducing slack variables [37], using tighter bounds, adding triple integral terms on the LK functional [161] are available in literature.

1.4.2 Stabilization of systems with input delay

A input delay system can be represented as

$$\dot{x}(t) = A_0 x(t) + B_1 u(t) + B_2 u(t-h), \tag{1.34}$$

where h is the delay associated with the input to the system.

Input-delay occurs due to the transmission of measured information [100,173], to acquire the information needed for decision-making, to generate the control decisions and to execute these decisions [152]. This type of delay is common in all feedback control systems, e.g. network control systems, biological systems, power systems etc. The control design for such systems are challenging issues. In [24, 25], a feedback controller is designed to stabilize the systems with state delay by transforming it into an ordinary delay-free system and using the concept of spectral stabilizability. The methods used in [24, 25] is difficult to implement for time-varying delays or uncertainties as they require the unstable poles of the system be known exactly and also resulting controller is distributed in nature. In [17, 52, 69], methods are proposed to directly design memoryless stabilizing controller for uncertain systems with state and input delays. Since this controller is independent of the delay, it tends to be unduly conservative, especially when the actual delay is small. The results are available in the literature [14, 34, 173, 179] on stabilization problem using static state feedback controller (i.e. memoryless controller). In [110, 111], a reduction method is proposed and the robust stabilization criterion is developed in the form of LMI. To fill the gap of reduction method in [110], a new state transformation approach is proposed in [173]. Under the new state transformation, the controller design only require to know the change of interval of the input delays rather than the exact values. Apart from the approaches discussed above, based on the state transformation, a descriptor system approach is applied in [14] to design robust controller. Though the stabilization criterion developed in [14] is less conservative but the controller structure includes integral control action which is difficult to implement.

1.4.3 Stabilization of systems using artificial delay

A stable feedback system without delay may become unstable with some delays [107]; whereas judicious introduction of delay may stabilize an unstable system [117,152,155]. For example, in metal milling machines, spindle speed is appropriately adjusted to achieve chattering stability [152] and in supply chain management, by adding delays for decision-making may be benefitted for purchasing and stocking decision [157]. The approach is otherwise known as wait-and-act control strategy [156].

The above discussions on existing approaches leads to the below motivations for this thesis.

1.5 Motivations of the present work

The review undertaken leads to some open problems that appears not to have been addressed so far in literature. These are as follows.

- The existing discretization/decomposition techniques (Section 1.3.4.1.A) are based on infinite-dimensional quadratic LK functional which leads to large-dimensional LMI with increasing intervals in discretizations.

- The available discretization/decomposition approaches (Section 1.3.4.1.A) for system with single delay can not be easily extended for control and filtering problems.

- Existing attempts for stability analysis of systems with multiple time-delays are mere extensions of single delay approaches as the delays are treated individually in analysis. The number of decision variables increases in the LMI with involvement of the number of delays in the system description.

- Robust Controller design for input-delay system is a challenging issue. Available techniques such as reduction technique, descriptor system approach using static state feedback controller are conservative.

- Use of delay in the controller for improving stabilizing ability is less investigated.

1.6 Scope of the thesis

The thesis covers the below issues.

- This work proposes a new stability criterion for systems with single delay using a simple delay-decomposition criterion.
- A new approach is proposed to investigate the stability analysis of systems with two-delays by exploiting overlapping information on them.
- The simple delay-decomposition approach is used to design a stabilizing controller for systems with single delay.
- A simple P-and PI-type stabilization criteria are developed for systems with input-delay.
- Stabilizing ability of a delayed dynamic state feedback controller is investigated.
- This thesis does not include any work on time-varying delay systems because the proposed approaches (delay-discretization approach for single delay and overlapping approach for two-delay system) in this thesis for time-delay systems are only applicable for constant delay cases.

1.7 Organization of the thesis

The present chapter briefly presents a review on stability and stabilization of time-delay systems. Remaining of the thesis is organized as follows.

- In **Chapter 2**, a delay-dependent stability criterion is proposed for systems with single delay by decomposing the delay interval. By defining a simple Lyapunov-Krasovskii functional for each of the decomposed intervals, a criterion is derived in such a way that a single one satisfies stability requirements of all the intervals. The same approach is extended to robust stability analysis problem.
- A delay-dependent stability criterion for systems with two constant delays is proposed in **Chapter 3**. As the delays are very small in nature, their ranges may overlap each other. This overlapping feature of the delays is exploited to derive a less conservative stability criterion. The same approach is extended to robust stability analysis as well.

- In **Chapter 4**, the delay-decomposition based approach using simple LK-functional proposed in Chapter 2 for stability analysis is used for developing static state-feedback stabilization criterion for systems with constant delay. During the development of the stabilization criterion in form of an LMI, non-linear terms comes up due to unknown controller parameter (K). To linearize such non-linear terms, a simple linearization approach is used. The simple stabilization criterion is further extended to robust stabilization criterion.

- **Chapter 5** deals with the stabilization of systems with input-delay using state-feedback controllers. First, a simple static state-feedback controller is used to derive stabilizing condition in the form of LMI. To formulate the LMI condition, a suitable LK functional is chosen and free-weighted matrix variable approach is employed. Next, to obtain a less conservative stabilization criterion using a simple static state-feedback controller, the delay-decomposition approach proposed in Chapter 2 is used. Finally, a PI-type controller is used to improve delay bound for the systems with input-delay.

- In **Chapter 6**, stabilizing ability of a class of delayed dynamic feedback controllers is investigated. A study is made on the improvement of the maximum tolerable delay bound by adding dynamicness in the controller and delay in controller state. To design such stabilizing controllers, a continuous pole placement technique (CPPT) for time-delay systems is used.

- **Chapter 7** concludes the thesis and proposes some future research directions led by the present work.

CHAPTER 2

Stability analysis for systems with single delay

Stability analysis is a major issue for a time-delay system. This chapter considers the problem of stability analysis of a system with single delay. A new simple delay-dependent stability criterion is proposed for stability analysis of linear time-delay systems with constant delay by decomposing the delay interval. On defining simple Lyapunov-Krasovskii functional for each of the decomposed intervals, a criterion is derived in such a way that a single one satisfies stability requirements of all the intervals. The development yields a simple, computationally efficient yet less conservative criterion than existing results. The same approach has also been extended for robust stability analysis. Numerical examples are presented to show the effectiveness of the development.

2.1 Introduction

The problem of reducing conservatism entails in finite-dimensional techniques to asses the stability of linear system with single delay has got a lot of attention in past decades [65, 66, 145–147]. Numerical discretization of infinite-dimensional complete quadratic LK functional exists whose numerical solution tracks the analytical result [43]. On the other hand, simple LK functional based approaches are more numerically attractive and easily extendable to control and filtering problems. However, such approaches are conservative due to use of simple LK functional and approximating certain integral terms while obtaining stability criterion. One way of reducing conservatism in such simple delay-dependent analysis is by decomposing the delay interval for reducing the so called gap in bounding the integral terms [9]. This behavior has already been exploited in [40] to obtain less conservative results while some extensions to control and filtering problems have been attempted in [8]. However, similar to the discretized approach of [43], the number of decision variables in this approach increases with increasing number of delay-decomposition.

To this end, along with the delay-dependent stability analysis, a delay-interval-dependent approach has been developed that appears to guarantee stability when the delay ranges in an interval [53]. This appears to be delay-interval like, since the stability criteria are derived using the information of non-zero lower bound of the uncertain delay term, and the tolerable delay upper bound increases with increase in lower bound. However, it is apparent that even the maximum tolerable delay bound achievable using such approaches for all possible lower limits is yet conservative to the analytical results available for constant delay case and, also, such criteria do not work on systems that are not stable for zero delay.

In this chapter, an improved delay-dependent stability criterion using a new multiple LK based approach is developed for stability analysis of linear time-delay systems. Simple LK functionals similar to delay-interval like analysis are defined over arbitrary number of decomposed delay intervals and it is shown that the criterion for the highest interval satisfies the stability requirement of all its sub-intervals leading to a single criterion that satisfies stability requirement for all the intervals in a shot. The criterion obtained this way is having same number of decision variables as that of considering single interval and hence is invariant to the number of intervals. Moreover, being based on the simple LK approach, the analysis may be easily extended to other problems involving time-delays. Such an extension to the robust analysis of time-delay systems is also presented. Numerical examples are presented demonstrating less conservativeness of this approach as well.

2.2 System description and preliminaries

Consider a linear time-delay system given by

$$\dot{x}(t) = Ax(t) + A_h x(t-h), \tag{2.1}$$

where $x(t) \in \Re^n$ is the system state. A and A_h are appropriate dimensional matrices, h is a constant delay satisfying $0 \le h_1 \le h \le h_2$ and $\bar{h} = h_2 - h_1$. Let us define $x_t = \{x(t) : t \in [-\bar{h} \quad 0]\}$. The initial condition for system (2.1), x_0 is first order differentiably smooth so that $\dot{x}_0$ exists and continuous.

Such a system is very much common in feedback control system, network control system, physiological system, economical system and so on. As it is well known that the delay has an effect on stability of the above systems, the stability analysis of such systems has got a lot of emphasis among the control communities. In view of lot of results available [9, 29, 31–34, 43, 44, 51, 63, 64, 164] on the stability analysis of system of type (2.1), some stability results are provided in the following sections. Before presenting the main result using proposed delay-decomposition approach, a simple delay-dependent stability criterion is proposed without using any delay-decomposition approach.

The following lemmas are required for the derivation of the results presented in this chapter.

Lemma 2.1 (Schur complement [6]). *For given constant matrices X_1, X_2 and X_3 of appropriate dimensions with invertible X_2, where $X_1^T = X_1$ and $X_2^T = X_2$, then*

$$X_1 + X_3^T X_2^{-1} X_3 < 0, \tag{2.2}$$

if and only if $\begin{bmatrix} X_1 & X_3^T \\ X_3 & -X_2 \end{bmatrix} < 0$ *or* $\begin{bmatrix} -X_2 & X_3 \\ X_3^T & X_1 \end{bmatrix} < 0$.

Lemma 2.2 ([130]). *For appropriate dimensional matrices X, Y and invertible matrix $Z > 0$, the following inequality holds:*

$$X^T Y + Y^T X \le X^T Z X + Y^T Z^{-1} Y.$$

2.3 Simple stability criterion

In this section, a simple LK functional based delay-dependent stability criterion is presented. To obtain the criterion, a tighter integral inequality condition i.e. Lemma 1.2 is used. Such results are already available in literature, e.g. please see [21].

Theorem 2.1. *System (2.1) is stable if there exists $P > 0$, $Q_k > 0$, $R_m > 0$, where $k = 1, \ldots, 4$, $m = 1, 2$, and arbitrary matrices M_j, N_j, satisfying the following LMI:*

$$\begin{bmatrix} \Pi & \bar{h}\Phi_j \\ * & -R_2 \end{bmatrix} < 0, \quad j = 1, 2, \tag{2.3}$$

where

$$\begin{aligned}
\Phi_1 &= \begin{bmatrix} 0 & N_1^T & M_1^T & 0 \end{bmatrix}^T, \Phi_2 = \begin{bmatrix} 0 & M_2^T & 0 & N_2^T \end{bmatrix}^T, \Pi = [\Pi_{ij}]_{i,j=1,\ldots,4}, \\
\Pi_{11} &= PA + A^T P + A^T \left\{ h_1^2 R_1 + \bar{h}^2 R_2 \right\} A + \sum_{k=1}^{3} Q_k - R_1, \\
\Pi_{12} &= PA_h + A^T \left\{ h_1^2 R_1 + \bar{h}^2 R_2 \right\} A_h, \Pi_{13} = R_1, \Pi_{14} = 0, \\
\Pi_{22} &= A_h^T \left\{ h_1^2 R_1 + \bar{h}^2 R_2 \right\} A_h - (Q_3 + Q_4) + \bar{h} \left(M_2 + M_2^T - N_1 - N_1^T \right), \\
\Pi_{23} &= \bar{h}(-M_1^T + N_1), \Pi_{24} = \bar{h}(-M_2 + N_2^T), \Pi_{33} = Q_4 - Q_2 - R_1 + \bar{h}(M_1 + M_1^T), \\
\Pi_{34} &= 0, \Pi_{44} = -Q_1 - \bar{h}(N_2 + N_2^T).
\end{aligned}$$

Proof. Consider a simple LK functional [146] as:

$$\begin{aligned}
V(x_t, \dot{x}_t) = x^T(t)Px(t) &+ \int_{t-h_2}^{t} x^T(\theta)Q_1 x(\theta)d\theta + \int_{t-h_1}^{t} x^T(\theta)Q_2 x(\theta)d\theta \\
+ \int_{t-h}^{t} x^T(\theta)Q_3 x(\theta)d\theta &+ \int_{t-h}^{t-h_1} x^T(\theta)Q_4 x(\theta)d\theta + h_1 \int_{t-h_1}^{t} \int_{\theta}^{t} \dot{x}^T(\phi)R_1 \dot{x}(\phi)d\phi d\theta \\
&+ \bar{h} \int_{t-h_2}^{t-h_1} \int_{\theta}^{t} \dot{x}^T(\phi)R_2 \dot{x}(\phi)d\phi d\theta.
\end{aligned} \tag{2.4}$$

The time-derivative of $V(x_t, \dot{x}_t)$ along the state trajectory of (2.1) is

$$\begin{aligned}
\dot{V}(x_t, \dot{x}_t) &= 2x^T(t)PAx(t) + 2x^T(t)PA_h x(t-h) + \sum_{k=1}^{3} x^T(t)Q_k x(t) \\
-x^T(t-h_1)(Q_2 - Q_4)x(t-h_1) &- \sum_{k=3}^{4} x^T(t-h)Q_k x(t-h) - x^T(t-h_2)Q_1 x(t-h_2) \\
+\dot{x}^T(t) \left\{ h_1^2 R_1 + \bar{h}^2 R_2 \right\} \dot{x}(t) &- h_1 \int_{t-h_1}^{t} \dot{x}^T(\theta)R_1 \dot{x}(\theta)d\theta - \bar{h} \int_{t-h_2}^{t-h_1} \dot{x}^T(\theta)R_2 \dot{x}(\theta)d\theta.
\end{aligned} \tag{2.5}$$

Following Lemma 1.2, the integral term in (2.5) satisfies

$$-h_1\int_{t-h}^{t}\dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta \le \begin{bmatrix} x(t) \\ x(t-h_1)\end{bmatrix}\begin{bmatrix} -R_1 & R_1 \\ * & -R_1\end{bmatrix}\begin{bmatrix} x(t) \\ x(t-h_1)\end{bmatrix}. \tag{2.6}$$

Last term in (2.5) may be written as:

$$-\bar{h}\int_{t-h_2}^{t-h_1}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta = -\bar{h}\int_{t-h}^{t-h_1}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta - \bar{h}\int_{t-h_2}^{t-h}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta. \tag{2.7}$$

One may approximate the above terms following Lemma 1.2 as:

$$\begin{aligned} -\int_{t-h}^{t-h_1}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta \quad &\le \begin{bmatrix} x(t-h_1) \\ x(t-h)\end{bmatrix}^T \left\{\begin{bmatrix} M_1+M_1^T & -M_1+N_1^T \\ * & -N_1-N_1^T\end{bmatrix}\right. \\ &\left. +\bar{h}\rho\begin{bmatrix} M_1 \\ N_1\end{bmatrix} R_2^{-1}\begin{bmatrix} M_1 \\ N_1\end{bmatrix}^T\right\}\begin{bmatrix} x(t-h_1) \\ x(t-h)\end{bmatrix}. \end{aligned} \tag{2.8}$$

$$\begin{aligned} -\int_{t-h_2}^{t-h}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta \quad &\le \begin{bmatrix} x(t-h) \\ x(t-h_2)\end{bmatrix}^T \left\{\begin{bmatrix} M_2+M_2^T & -M_2+N_2^T \\ * & -N_2-N_2^T\end{bmatrix}\right. \\ &\left. +\bar{h}(1-\rho)\begin{bmatrix} M_2 \\ N_2\end{bmatrix} R_2^{-1}\begin{bmatrix} M_2 \\ N_2\end{bmatrix}^T\right\}\begin{bmatrix} x(t-h) \\ x(t-h_2)\end{bmatrix}. \end{aligned} \tag{2.9}$$

Substituting (2.6), (2.8) and (2.9) into (2.5), one may write

$$\dot{V}(x_t,\dot{x}_t) \le \xi^T(t)(\Pi + \rho\bar{h}^2\Phi_1R_2^{-1}\Phi_1^T + (1-\rho)\bar{h}^2\Phi_2R_2^{-1}\Phi_2^T)\xi(t), \tag{2.10}$$

where

$$\xi(t) \quad = \quad \begin{bmatrix} x^T(t) & x^T(t-h) & x^T(t-h_1) & x^T(t-h_2)\end{bmatrix}^T; \rho = \frac{h-h_1}{\bar{h}}, \quad 0 \le \rho \le 1;$$

and Φ_1, Φ_2 are as given in (2.3). Therefore, the stability requirement for (2.1) is

$$\Pi + \rho\bar{h}^2\Phi_1R_2^{-1}\Phi_1^T + (1-\rho)\bar{h}^2\Phi_2R_2^{-1}\Phi_2^T < 0. \tag{2.11}$$

The above is a polytope of matrices on ρ and is always negative definite if its two certain vertices are negative definite. Then, (2.11) can be written as:

$$\Pi + \bar{h}^2 \Phi_j R_2^{-1} \Phi_j^T < 0, \quad j = 1, 2. \tag{2.12}$$

One can write, (2.12) as:

$$\Pi + (\bar{h}\Phi_j) R_2^{-1} (\bar{h}\Phi_j)^T < 0, \quad j = 1, 2. \tag{2.13}$$

Following Lemma 2.1, and taking Schur complement once, one obtains (2.3). This completes the proof. □

Theorem 2.1 presents a delay-interval-dependent stability criterion for system (2.1) without decomposing the delay interval in line of work [21, 146]. In the next section, a delay-decomposition approach is used to derive an improved stability criterion for (2.1).

2.4 Stability criterion using delay-decomposition

Stability criteria using the existing delay-decomposition approaches results in infinite-dimensional LMI condition which are difficult to solve with available tools [43, 51]. Being motivated to fill the gap of the available delay-decomposition techniques, a new technique is proposed in which the tolerable delay range $\bar{h}$ divided into N number of δ intervals of equal measure so that one may define

$$h_i = \begin{cases} 0 & \text{for} \quad i = 0, \\ i\delta & \text{for} \quad i = 1, 2, \ldots, N-1, \\ \bar{h} & \text{for} \quad i = N. \end{cases} \tag{2.14}$$

The main objective of this work is to derive a stability criterion for (2.1) by adopting a new multiple LK functional approach based on decomposition of the total delay interval. The following theorem presents an LMI based stability criterion for analyzing stability of system (2.1).

Theorem 2.2. *System (2.1) with $h_1 = 0$, with given h_2 and δ conforming (2.14), is stable if there exists $P > 0$, $Q_k > 0$, $R_m > 0$, where $k = 1, \ldots, 4$, $m = 1, 2$ and arbitrary matrices*

M_j, N_j, satisfying the following LMI:

$$\begin{bmatrix} \Theta & \delta\Phi_j \\ * & -R_2 \end{bmatrix} < 0, \quad j = 1, 2, \tag{2.15}$$

where

$$\begin{aligned}
\Phi_1 &= \begin{bmatrix} 0 & N_1^T & M_1^T & 0 \end{bmatrix}^T, \Phi_2 = \begin{bmatrix} 0 & M_2^T & 0 & N_2^T \end{bmatrix}^T, \Theta = [\Theta_{ij}]_{i,j=1,\ldots,4}, \\
\Theta_{11} &= PA + A^T P + A^T \left\{ (\bar{h}-\delta)^2 R_1 + \delta^2 R_2 \right\} A + \sum_{k=1}^{3} Q_k - R_1, \\
\Theta_{12} &= PA_h + A^T \left\{ (\bar{h}-\delta)^2 R_1 + \delta^2 R_2 \right\} A_h, \Theta_{13} = R_1, \Theta_{14} = 0, \\
\Theta_{22} &= A_h^T \left\{ (\bar{h}-\delta)^2 R_1 + \delta^2 R_2 \right\} A_h - (Q_3 + Q_4) + \delta \left(M_2 + M_2^T - N_1 - N_1^T \right), \\
\Theta_{23} &= \delta(-M_1^T + N_1), \Theta_{24} = \delta(-M_2 + N_2^T), \Theta_{33} = Q_4 - Q_2 - R_1 + \delta(M_1 + M_1^T), \\
\Theta_{34} &= 0, \Theta_{44} = -Q_1 - \delta(N_2 + N_2^T).
\end{aligned}$$

Proof. Considering the i^{th} interval that $h \in [h_{(i-1)}, h_i]$, a simple LK functional is defined as [146]:

$$\begin{aligned}
V_i(x_t, \dot{x}_t) &= x^T(t)Px(t) + \sum_{j=1}^{2} \int_{t-h_{(i+1-j)}}^{t} x^T(\theta)Q_j x(\theta)d\theta + \int_{t-h}^{t} x^T(\theta)Q_3 x(\theta)d\theta \\
&+ \int_{t-h}^{t-h_{(i-1)}} x^T(\theta)Q_4 x(\theta)d\theta + h_{(i-1)} \int_{t-h_{(i-1)}}^{t} \int_{\theta}^{t} \dot{x}^T(\phi)R_1\dot{x}(\phi)d\phi d\theta \\
&+\delta \int_{t-h_i}^{t-h_{(i-1)}} \int_{\theta}^{t} \dot{x}^T(\phi)R_2\dot{x}(\phi)d\phi d\theta.
\end{aligned} \tag{2.16}$$

Differentiating $V_i(x_t, \dot{x}_t)$ with respect to time along the state trajectory of (2.1) yields

$$\begin{aligned}
\dot{V}_i(x_t, \dot{x}_t) &= 2x^T(t)PAx(t) + 2x^T(t)PA_h x(t-h) + \sum_{k=1}^{3} x^T(t)Q_k x(t) \\
&-x^T(t-h_{(i-1)})(Q_2 - Q_4)x(t-h_{(i-1)}) - \sum_{k=3}^{4} x^T(t-h)Q_k x(t-h) - x^T(t-h_i)Q_1 x(t-h_i) \\
&+\dot{x}^T(t) \left\{ h_{(i-1)}^2 R_1 + \delta^2 R_2 \right\} \dot{x}(t) - h_{(i-1)} \int_{t-h_{(i-1)}}^{t} \dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta - \delta \int_{t-h_i}^{t-h_{(i-1)}} \dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta.
\end{aligned} \tag{2.17}$$

Following Lemma 1.2, the first integral in (2.17) satisfies

$$-h_{(i-1)}\int\limits_{t-h_{(i-1)}}^{t}\dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta \leq \begin{bmatrix} x(t) \\ x(t-h_{(i-1)}) \end{bmatrix}\begin{bmatrix} -R_1 & R_1 \\ * & -R_1 \end{bmatrix}\begin{bmatrix} x(t) \\ x(t-h_{(i-1)}) \end{bmatrix}. \quad (2.18)$$

Last term in (2.17) may be written as:

$$-\delta\int\limits_{t-h_i}^{t-h_{(i-1)}}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta = -\delta\int\limits_{t-h}^{t-h_{(i-1)}}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta - \delta\int\limits_{t-h_i}^{t-h}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta. \quad (2.19)$$

One may approximate the above terms following Lemma 1.2 as:

$$\begin{aligned} -\int\limits_{t-h}^{t-h_{(i-1)}}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta \quad &\leq \begin{bmatrix} x(t-h_{(i-1)}) \\ x(t-h) \end{bmatrix}^T\left\{\begin{bmatrix} M_1+M_1^T & -M_1+N_1^T \\ * & -N_1-N_1^T \end{bmatrix}\right. \\ &\left. +\delta\rho\begin{bmatrix} M_1 \\ N_1 \end{bmatrix}R_2^{-1}\begin{bmatrix} M_1 \\ N_1 \end{bmatrix}^T\right\}\begin{bmatrix} x(t-h_{(i-1)}) \\ x(t-h) \end{bmatrix}. \end{aligned} \quad (2.20)$$

$$\begin{aligned} -\int\limits_{t-h_i}^{t-h}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta \quad &\leq \begin{bmatrix} x(t-h) \\ x(t-h_i) \end{bmatrix}^T\left\{\begin{bmatrix} M_2+M_2^T & -M_2+N_2^T \\ * & -N_2-N_2^T \end{bmatrix}\right. \\ &\left. +\delta(1-\rho)\begin{bmatrix} M_2 \\ N_2 \end{bmatrix}R_2^{-1}\begin{bmatrix} M_2 \\ N_2 \end{bmatrix}^T\right\}\begin{bmatrix} x(t-h) \\ x(t-h_i) \end{bmatrix}. \end{aligned} \quad (2.21)$$

Substituting (2.18), (2.22) and (2.21) into (2.17), one may write

$$\dot{V}_i(x_t,\dot{x}_t) \leq \xi^T(t)(\Psi + h_{(i-1)}^2\Omega_i + \rho\delta^2\Phi_1R_2^{-1}\Phi_1^T + (1-\rho)\delta^2\Phi_2R_2^{-1}\Phi_2^T)\xi(t), \quad (2.22)$$

where

$$\begin{aligned} \xi(t) &= \begin{bmatrix} x^T(t) & x^T(t-h) & x^T(t-h_{(i-1)}) & x^T(t-h_i) \end{bmatrix}^T; \Psi = [\Psi_{ij}]_{i,j=1,\ldots,4}, \\ \Psi_{11} &= PA + A^TP + \sum_{k=1}^{3}Q_k - R_1 + \delta^2A^TR_2A, \Psi_{12} = PA_h + \delta^2A^TR_2A_h, \\ \Psi_{13} &= R_1, \Psi_{14} = 0, \Psi_{22} = -(Q_3+Q_4) + \delta\left(M_2 + M_2^T - N_1 - N_1^T\right) + \delta^2A_h^TR_2A_h, \\ \Psi_{23} &= \delta(-M_1^T + N_1), \Psi_{24} = \delta(-M_2 + N_2^T), \Psi_{33} = Q_4 - Q_2 - R_1 + \delta(M_1 + M_1^T), \end{aligned}$$

$$\Psi_{34} = 0, \Psi_{44} = -Q_1 - \delta(N_2 + N_2^T);$$

$$\Omega_i = \begin{bmatrix} A^T R_1 A & A^T R_1 A_h & 0 & 0 \\ * & A_h^T R_1 A_h & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix}, \rho = \frac{h - h_{i-1}}{\delta}, \quad 0 \leq \rho \leq 1;$$

and Φ_1, Φ_2 are as given in (2.15). Therefore, the stability requirement for the i^{th} interval is

$$\Psi + h_{(i-1)}^2 \Omega_i + \rho\delta^2 \Phi_1 R_2^{-1} \Phi_1^T + (1-\rho)\delta^2 \Phi_2 R_2^{-1} \Phi_2^T < 0. \tag{2.23}$$

Then, (2.23) can be equivalently written as:

$$\Psi + h_{(i-1)}^2 \Omega_i + \delta^2 \Phi_j R_2^{-1} \Phi_j^T < 0, \quad j = 1, 2. \tag{2.24}$$

To this end, note that, $\Omega_i \geq 0$ and the term $h_{(i-1)}^2 \Omega_i$ is maximum when $h \in [h_{(N-1)}, \bar{h}]$, the N^{th} interval. Therefore, irrespective of h lies in any of the intervals, the following condition always ensures stability of (2.1):

$$\Psi + h_{(N-1)}^2 \Omega_N + \delta^2 \Phi_j R_2^{-1} \Phi_j^T < 0, \quad j = 1, 2. \tag{2.25}$$

Finally, taking Schur complement for the last term in (2.25), one obtains (2.15). □

Remark 2.1. *Unlike conventional discretization/decomposition approaches of [43, 51, 104], the number of decision variables and size of the LMI in Theorem 2.2 does not increase with N. It appears that no approximation is used in obtaining (2.25) from (2.24) to fetch this benefit. However, approximation of two integral inequalities are involved in the derivative of LK functional (2.17). Although the conservativeness of the approximation in the second one decreases with increase in number of delay decomposition (N), but the first one doesn't. This is because the final stability criterion is corresponding to the N^{th} interval of delay, for which the integral limit of the first integral in (2.17) is the largest compared to other cases. This introduces conservatism with large N. However, one may easily search over N to obtain the maximum tolerable $\bar{h}$.*

The stability criterion developed in Theorem 2.2 may be conservatively (for specific systems) simplified by eliminating the free-variables and reducing the dimension of the LMI. The following corollary presents this result.

Corollary 2.1. *System (2.1) with $h_1 = 0$ with given h_2 and δ conforming (2.14), is stable if there exist $P > 0$, $Q_k > 0$, $R_j > 0$, $k = 1, \ldots, 4$, $j = 1, 2$, satisfying the following LMI condition:*

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} & R_1 & 0 \\ * & \bar{\Theta}_{22} & R_2 & R_2 \\ * & * & \bar{\Theta}_{33} & 0 \\ * & * & * & \bar{\Theta}_{44} \end{bmatrix} < 0, \tag{2.26}$$

where Θ_{11} and Θ_{12} are as defined in (2.15) and

$$\begin{aligned} \bar{\Theta}_{22} &= -(Q_3 + Q_4) + A_h^T \left\{(\bar{h} - \delta)^2 R_1 + \delta^2 R_2\right\} A_h - 2R_2, \\ \bar{\Theta}_{33} &= Q_4 - Q_2 - R_1 - R_2, \bar{\Theta}_{44} = -Q_1 - R_2. \end{aligned}$$

Proof. Since the last term in (2.23) is positive definite, one may reduce the stability condition in the form of a single matrix inequality as:

$$\Psi + h_{(N-1)}^2 \Omega_N + \delta^2 \Phi_1 R_2^{-1} \Phi_1^T + \delta^2 \Phi_2 R_2^{-1} \Phi_2^T < 0. \tag{2.27}$$

one may write (2.27) as:

$$\Theta + \delta^2 \Phi_1 R_2^{-1} \Phi_1^T + \delta^2 \Phi_2 R_2^{-1} \Phi_2^T < 0. \tag{2.28}$$

Separating the M_1, N_1, M_2 and N_2 terms from Θ, one obtains

$$\Upsilon + (\delta\Phi_1) I_1^T + I_1 (\delta\Phi_1)^T + (\delta\Phi_1) R_2^{-1} (\delta\Phi_1)^T + (\delta\Phi_2) I_2^T + I_2 (\delta\Phi_2)^T + (\delta\Phi_2) R_2^{-1} (\delta\Phi_2)^T < 0, \tag{2.29}$$

where

$$\begin{aligned} \Upsilon &= [\Upsilon_{ij}]_{i,j=1,\ldots,4}, \Upsilon_{11} = PA + A^T P + \sum_{k=1}^{3} Q_k - R_1 + A^T \left\{(\bar{h} - \delta)^2 R_1 + \delta^2 R_2\right\} A, \\ \Upsilon_{12} &= PA_h + A^T \left\{(\bar{h} - \delta)^2 R_1 + \delta^2 R_2\right\} A_h, \Upsilon_{13} = R_1, \Upsilon_{14} = 0, \\ \Upsilon_{22} &= -\sum_{k=3}^{4} Q_k + A_h^T \left\{(\bar{h} - \delta)^2 R_1 + \delta^2 R_2\right\} A_h, \Upsilon_{23} = 0, \Upsilon_{24} = 0, \Upsilon_{33} = -(Q_2 - Q_4) - R_1, \\ \Upsilon_{34} &= 0, \Upsilon_{44} = -Q_1, I_1 = \begin{bmatrix} 0 & -I & I & 0 \end{bmatrix}^T, I_2 = \begin{bmatrix} 0 & I & 0 & -I \end{bmatrix}^T. \end{aligned}$$

One can write (2.29) as:

$$\Upsilon+(\delta\Phi_1 + I_1R_2)\, R_2^{-1}(\delta\Phi_1 + I_1R_2)^T - I_1R_2I_1^T + (\delta\Phi_2 + I_2R_2)\, R_2^{-1}(\delta\Phi_2 + I_2R_2)^T - I_2R_2I_2^T < 0. \tag{2.30}$$

Further, following Lemma 1.2, substituting the free variables as $M_i = {M_i}^T = -N_i = -{N_i}^T = -\delta^{-1}R_2$, the above stability condition yields (2.26). □

Remark 2.2. *Although the above criterion may be conservative in comparison with Theorem 2.2 due to the approximations incorporated, the gap between the two criteria decreases with decreasing the integral limit δ (increasing N) and hence more useful for larger N since it involves lesser no. of variables.*

To validate the above criteria, some well-known numerical examples are used for stability analysis, which are presented next:

2.4.1 Numerical examples

Two numerical examples are now considered to illustrate the effectiveness of the proposed approach.

Example 2.1. Consider system (2.1) with

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, A_h = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}.$$

The variation of delay bound $\bar{h}$ with number of interval N obtained using the Theorem 2.2 and Corollary 2.1 is shown in Fig. 2.1 (they overlap each other as they respond equally). It can be seen that, the maximum $\bar{h}$ obtained for $N = 2$ using the present approach is same as that obtained using available decomposition based approaches with two decompositions [8,40,51,128]. With further increase in N, the computed $\bar{h}$ decreases. Such a behavior can be followed from Remark 2.1. The derivative of the LK functional (2.17) involves two integral inequalities. At $N = 2$, integral limits of both the integral inequalities in (2.17) is halved and so the bounding gap reduces for both, leading to improved result. As N increases the conservativeness of the approximation in the second one decreases with increase in number of delay-decomposition (N), but the first one doesn't. Rather the conservativeness increases with increase in number of delay-decomposition. Therefore, one always gets maximum delay value at $N = 2$.

The maximum $\bar{h}$ obtained using Theorem 2.2 and Corollary 2.1 are tabulated in Table 2.1, along with some cursory existing results which shows that the present result considerably improves the computed $\bar{h}$.

Note that, in this example, eigenvalues of non-delayed state matrix (A) are -2 and -0.9

Table 2.1: Comparison of delay bound ($\bar{h}$) for Example 2.1

Methods	$\bar{h}$
[33], [53], [145], [146], [147],Theorem 2.1	4.472
[2]	5.120
Theorem 2.2 ($N = 2$)	5.717
Corollary 2.1 ($N = 2$)	5.717
Analytical	6.172

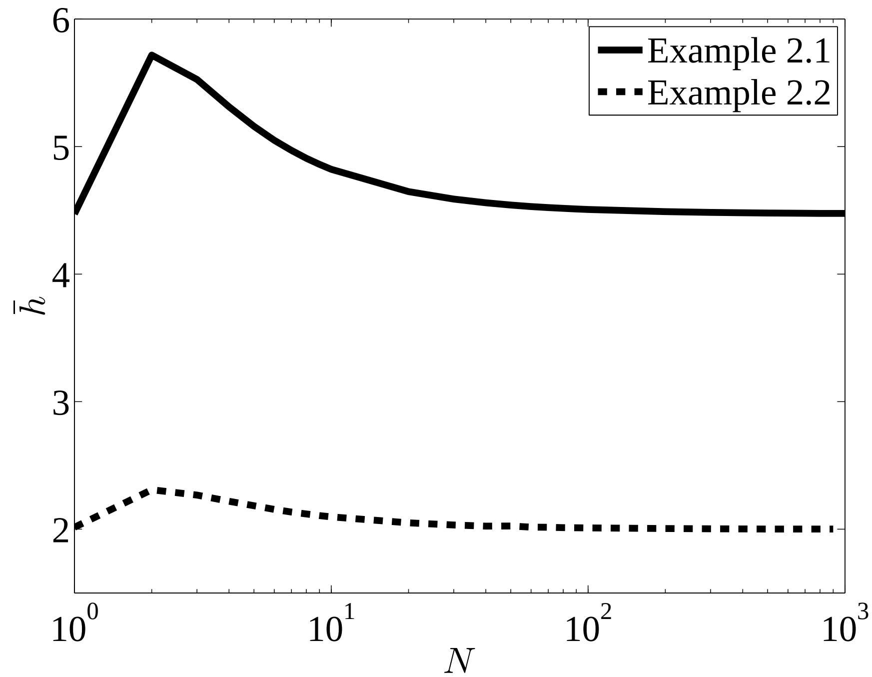

Figure 2.1: Variation of N w.r.t number of interval for Example 2.1 and 2.2

and eigenvalues of delayed state matrix (A_h) are -1 and -1. Hence, both the matrices are stable. The proposed delay-decomposition approach is used to obtain the delay margin for the system. Another example is considered that is having A_h with a positive real eigenvalue but the system is again delay-dependently stable.

Example 2.2. Consider another example of (2.1) with

$$A = \begin{bmatrix} -3 & -2.5 \\ 1 & 0.5 \end{bmatrix}, A_h = \begin{bmatrix} 1.5 & 2.5 \\ -0.5 & -1.5 \end{bmatrix}.$$

For this system, maximum $\bar{h}$ obtained using Theorem 2.2 and Corollary 2.1 is tabulated in Table 2.2 and the variation $\bar{h}$ with number of interval N is shown in Fig. 2.1. It can be observed that in this case also maximum delay bound is obtained at $N = 2$ as expected.

Table 2.2: Comparison of delay bound ($\bar{h}$) for Example 2.2

Methods	$\bar{h}$
[145], [146]	1.9998
[147],Theorem 2.1	2.0050
Theorem 2.2 ($N = 2$)	2.3094
Corollary 2.1 ($N = 2$)	2.3094
Analytical	2.4184

For this example, non-delayed state matrix (A) are -2 and -0.5 and eigen values of system delayed state matrix (A_h) are 1 and -1. The delayed states are unstable. For this case also, the approach is observed to be less conservative than that existing results. Both the examples demonstrate that the proposed method yields quite less conservative results compared to the existing ones. It may also be noted that computational complexity is similar to those existing approaches since the result is based on a simple LMI that corresponds to complexity involved with considering the whole delay as a single interval.

2.5 Robust analysis using delay-decomposition

Parametric variations exist in physical systems due to approximation, ignored factors, presence of disturbance or noise and so on. Therefore, stability analysis for nominal system may not work for the corresponding actual system. At the same time, the presence of delay makes the analysis more difficult and challenging. This motivates the researcher to work on robust stability analysis of time-delay systems. A considerable number of results are available on such analysis. Some major contributions are highlighted here. The robust stability analysis of time-delay systems with norm-bounded uncertainty is considered in [89, 90] using LR approach. The delay-dependent robust stability condition is investigated in [68] where bounded inequalities are used. A descriptor model transformation is introduced in [32] for robust stability analysis. To obtain less conservative criteria, the idea of relaxation with free-slack matrices is introduced in [167]. A free-weighted matrix approach is proposed in [55] to obtain a less conservative criterion. The same approach is widely used to obtain less conservative results [94, 97]. The stability problem of uncertain system using complete quadratic Lyapunov functional is investigated in [41,42]. The decomposition technique proposed in [41,42] generates a infinite-dimensional LMI condition for robust analysis. To handle such issue, the decomposition technique using simple LK functional proposed in the previous section can be extended for the case of robust analysis.

Consider an uncertain time-delay system with norm-bounded uncertainties described as:

$$\dot{x}(t) = \bar{A}(t)x(t) + \bar{A}_h(t)x(t-h), \tag{2.31}$$

where $\bar{A}(t) = (A + \Delta A(t))$, $\bar{A}_h(t) = (A_h + \Delta A_h(t))$, A and A_h are constant matrices of appropriate dimensions, $\Delta A(t)$ and $\Delta A_h(t)$ are unknown matrices representing time-varying norm-bounded uncertainties and can be decomposed as:

$$\Delta A(t) = D_a F_a(t) E_a, \Delta A_h(t) = D_h F_h(t) E_h, \tag{2.32}$$

where $F_a(t)$ and $F_h(t)$ are time-varying matrices satisfying

$$F_a^T(t)F_a(t) \leq I, F_h^T(t)F_h(t) \leq I, \tag{2.33}$$

and D_a, D_h, E_a and E_h are constant matrices.
As it is observed in previous section that, despite being approximate and simple, Corollary 2.1 yields almost same results as obtained using Theorem 2.2 with larger N, Corollary 2.1 is used for obtaining stability criteria for (2.31) since less number of free-variables involved in it.

Theorem 2.3. *System (2.31) with $h_1 = 0$ with given h_2 and δ is stable conforming (2.14), if there exist $P > 0$, $Q_k > 0$, $R_j > 0$, where $k = 1, \ldots, 4$, and $j = 1, 2$, satisfying the following*

LMI:

$$\begin{bmatrix} \Sigma & \Gamma \\ * & \varepsilon \end{bmatrix} < 0, \tag{2.34}$$

where

$$\begin{aligned}
\Sigma &= [\Sigma_{ij}]_{i,j=12\ldots6.}, \Sigma_{11} = PA + A^T P + \sum_{i=1}^{3} Q_i - R_1 + \varepsilon_1 E_a^T E_a, \Sigma_{12} = PA_h, \Sigma_{13} = R_1, \\
\Sigma_{14} &= 0, \Sigma_{15} = (\bar{h} - \delta)A^T R_1, \Sigma_{16} = \delta A^T R_2, \Sigma_{22} = -Q_3 - Q_4 - 2R_2 + \varepsilon_2 E_h^T E_h^T, \Sigma_{23} = R_2, \\
\Sigma_{24} &= R_2, \Sigma_{25} = (\bar{h} - \delta)A_h^T R_1, \Sigma_{26} = \delta A_h^T R_2, \Sigma_{33} = Q_4 - Q_2 - R_1 - R_2, \Sigma_{34} = 0, \Sigma_{35} = 0, \\
\Sigma_{36} &= 0, \Sigma_{44} = -Q_1 - R_2, \Sigma_{45} = 0, \Sigma_{46} = 0, \Sigma_{55} = -R_1, \Sigma_{56} = 0, \Sigma_{66} = -R_2, \\
\Gamma &= \begin{bmatrix} D_a^T P & 0 & 0 & 0 & (\bar{h} - \delta)D_a^T R_1 & \delta D_a^T R_2 \\ D_h^T P & 0 & 0 & 0 & (\bar{h} - \delta)D_h^T R_1 & \delta D_h^T R_2 \end{bmatrix}^T, \varepsilon = diag\{-\varepsilon_a I, -\varepsilon_h I\}.
\end{aligned}$$

Proof. Following Corollary 2.1, stability of (2.31) can be ensured if the following is satisfied:

$$\Sigma(t) < 0, \tag{2.35}$$

where

$$\begin{aligned}
\Sigma(t) &= \begin{bmatrix} \Sigma_{11}(t) & \Sigma_{12}(t) & R_1 & 0 \\ * & \Sigma_{22}(t) & R_2 & R_2 \\ * & * & \bar{\Theta}_{33} & 0 \\ * & * & * & \bar{\Theta}_{44} \end{bmatrix}, \\
\Sigma_{11}(t) &= P\bar{A}(t) + \bar{A}^T(t)P + \sum_{i=1}^{3} Q_i + \bar{A}^T(t)\bar{R}\bar{A}(t) - R_1, \Sigma_{12}(t) = PA_h(t) + \bar{A}^T(t)\bar{R}\bar{A}_h(t), \\
\Sigma_{22}(t) &= -Q_3 - Q_4 + \bar{A}_h^T(t)\bar{R}\bar{A}_h(t) - 2R_2, \bar{R} = \left\{ \left(\bar{h} - \delta\right)^2 R_1 + \delta^2 R_2 \right\}.
\end{aligned}$$

Taking Schur complement twice, in order to linearize the quadratic uncertain terms in $\Sigma_{11}(t)$, $\Sigma_{12}(t)$ and $\Sigma_{22}(t)$ above, (2.35) may be written as:

$$\tilde{\Sigma}(t) = \begin{bmatrix} \tilde{\Sigma}_{11}(t) & \tilde{\Sigma}_{12}(t) & R_1 & 0 & \tilde{\Sigma}_{15}(t) & \tilde{\Sigma}_{16}(t) \\ * & \tilde{\Sigma}_{22} & R_2 & R_2 & \tilde{\Sigma}_{25}(t) & \tilde{\Sigma}_{26}(t) \\ * & * & \bar{\Theta}_{33} & 0 & 0 & 0 \\ * & * & * & \bar{\Theta}_{44} & 0 & 0 \\ * & * & * & * & -R_1 & 0 \\ * & * & * & * & * & -R_2 \end{bmatrix} < 0, \qquad (2.36)$$

where

$$\begin{aligned} \tilde{\Sigma}_{11}(t) &= P\bar{A}(t) + \bar{A}^T(t)P + \sum_{i=1}^{3} Q_i - R_1, \tilde{\Sigma}_{12}(t) = P\bar{A}_h(t), \tilde{\Sigma}_{15}(t) = (\bar{h}-\delta)\bar{A}^T(t)R_1, \\ \tilde{\Sigma}_{16}(t) &= \delta\bar{A}^T(t)R_2, \tilde{\Sigma}_{22} = -Q_3 - Q_4 - 2R_2, \tilde{\Sigma}_{25}(t) = (\bar{h}-\delta)\bar{A}_h{}^T(t)R_1, \tilde{\Sigma}_{26}(t) = \delta\bar{A}_h{}^T(t)R_2. \end{aligned}$$

Separating the uncertain terms, (2.36) can be written as:

$$\tilde{\Sigma}(t) = \bar{\Sigma} + \hat{\Sigma}(t) < 0, \qquad (2.37)$$

where

$$\begin{aligned} \bar{\Sigma} &= \begin{bmatrix} \bar{\Sigma}_{11} & PA_h & R_1 & 0 & \bar{\Sigma}_{15} & \bar{\Sigma}_{16} \\ * & \tilde{\Sigma}_{22} & R_2 & R_2 & \bar{\Sigma}_{25} & \bar{\Sigma}_{26} \\ * & * & \bar{\Theta}_{33} & 0 & 0 & 0 \\ * & * & * & \bar{\Theta}_{44} & 0 & 0 \\ * & * & * & * & -R_1 & 0 \\ * & * & * & * & * & -R_2 \end{bmatrix}, \\ \hat{\Sigma}(t) &= \begin{bmatrix} \hat{\Sigma}_{11}(t) & \hat{\Sigma}_{12}(t) & 0 & 0 & \hat{\Sigma}_{15}(t) & \hat{\Sigma}_{16}(t) \\ * & 0 & 0 & 0 & \hat{\Sigma}_{25}(t) & \hat{\Sigma}_{26}(t) \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix}, \\ \bar{\Sigma}_{11} &= PA + A^TP + \sum_{i=1}^{3} Q_i - R_1, \bar{\Sigma}_{15} = (\bar{h}-\delta)A^TR_1, \bar{\Sigma}_{16} = \delta A^TR_2, \\ \bar{\Sigma}_{25} &= (\bar{h}-\delta)A_h^TR_1, \bar{\Sigma}_{26} = \tau A_h^TR_2, \hat{\Sigma}_{11}(t) = P\Delta A + \Delta A^TP, \hat{\Sigma}_{12}(t) = P\Delta A_h, \\ \hat{\Sigma}_{15}(t) &= (\bar{h}-\delta)\Delta A^TR_1, \hat{\Sigma}_{16}(t) = \delta\Delta A^TR_2, \hat{\Sigma}_{25}(t) = (\bar{h}-\delta)\Delta A_h^TR_1, \hat{\Sigma}_{26}(t) = \delta\Delta A_h^TR_2. \end{aligned}$$

In view of (2.33), the uncertain matrix $\hat{\Sigma}(t)$ may be decomposed as

$$\hat{\Sigma}(t) = X_a^T F_a^T(t) Y_a + Y_a^T F_a(t) X_a + X_h^T F_h^T(t) Y_h + Y_h^T F_h(t) X_h, \tag{2.38}$$

where

$$\begin{aligned}
X_a &= \begin{bmatrix} E_a & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, X_h = \begin{bmatrix} 0 & E_h & 0 & 0 & 0 & 0 \end{bmatrix}, \\
Y_a &= \begin{bmatrix} D_a^T P & 0 & 0 & 0 & (\bar{h}-\delta) D_a^T R_1 & \delta D_a^T R_2 \end{bmatrix}, \\
Y_h &= \begin{bmatrix} D_h^T P & 0 & 0 & 0 & (\bar{h}-\delta) D_h^T R_1 & \delta D_h^T R_2 \end{bmatrix}.
\end{aligned}$$

By following Lemma 2.2, one can easily obtain

$$\begin{aligned}
X_a^T F_a^T(t) Y_a + Y_a^T F_a^T(t) X_a &\leq \varepsilon_a X_a^T X_a + \varepsilon_a^{-1} Y_a^T Y_a, \\
X_h^T F_h^T(t) Y_h + Y_h^T F_h^T(t) X_h &\leq \varepsilon_h X_h^T X_h + \varepsilon_h^{-1} Y_h^T Y_h.
\end{aligned}$$

From the above inequalities, one obtains

$$\varepsilon_a X_a^T X_a + \varepsilon_h X_h^T X_h + \varepsilon_a^{-1} Y_a^T Y_a + \varepsilon_h^{-1} Y_h^T Y_h \geq \hat{\Sigma}(t). \tag{2.39}$$

Then, (2.37) can be written as

$$\Sigma + \varepsilon_a^{-1} Y_a^T Y_a + \varepsilon_h^{-1} Y_h^T Y_h < 0, \tag{2.40}$$

where Σ is defined in (2.34). Finally, taking Schur complement, one obtains (2.34). □

In some cases, the uncertainty is described in (2.32) may be decomposed in simple fashion with $D_a = D_h = D$ and $F_a(t) = F_h(t) = F(t)$. In such cases, the above analysis may be conservative due to individual treatment of uncertain terms in the analysis. For the case $D_a = D_h = D$ and $F_a(t) = F_h(t) = F(t)$, some benefits may be extracted by treating two uncertain terms ΔA and ΔA_h conjugatively. The following corollary utilizes this treatment.

Corollary 2.2. *The system (2.31) with $h_1 = 0$ with given h_2 and δ is stable conforming (2.14), if there exist $P > 0$, $Q_k > 0$, $R_j > 0$, where $k = 1, \ldots, 4$, and $j = 1, 2$, satisfying the following LMI:*

$$\begin{bmatrix} \breve{\Sigma} & \breve{\Gamma} \\ * & -\varepsilon I \end{bmatrix} < 0, \tag{2.41}$$

where

$$
\begin{aligned}
\breve{\Sigma} &= [\breve{\Sigma}_{ij}]_{i,j=12\ldots 6.}, \breve{\Sigma}_{11} = PA + A^T P + \sum_{i=1}^{3} Q_i - R_1 + \varepsilon E_a^T E_a, \breve{\Sigma}_{12} = PA_h + \varepsilon E_a^T E_h, \\
\breve{\Sigma}_{13} &= R_1, \breve{\Sigma}_{14} = 0, \breve{\Sigma}_{15} = (\bar{h} - \delta) A^T R_1, \breve{\Sigma}_{16} = \delta A^T R_2, \breve{\Sigma}_{22} = -Q_3 - Q_4 - 2R_2 + \varepsilon E_h^T E_h^T, \\
\breve{\Sigma}_{23} &= R_2, \breve{\Sigma}_{24} = R_2, \breve{\Sigma}_{25} = (\bar{h} - \delta) A_h^T R_1, \breve{\Sigma}_{26} = \delta A_h^T R_2, \breve{\Sigma}_{33} = Q_4 - Q_2 - R_1 - R_2, \\
\breve{\Sigma}_{34} &= 0, \breve{\Sigma}_{35} = 0, \breve{\Sigma}_{36} = 0 \breve{\Sigma}_{44} = -Q_1 - R_2, \breve{\Sigma}_{45} = 0, \breve{\Sigma}_{46} = 0, \breve{\Sigma}_{55} = -R_1, \breve{\Sigma}_{56} = 0, \\
\breve{\Sigma}_{66} &= -R_2, \breve{\Gamma} = \begin{bmatrix} D^T P & 0 & 0 & 0 & (\bar{h} - \delta) D^T R_1 & \delta D^T R_2 \end{bmatrix}^T.
\end{aligned}
$$

Proof. In view of (2.33), the uncertain matrix $\hat{\Sigma}(t)$ may be decomposed as

$$
\hat{\Sigma}(t) = X^T F^T(t) \breve{\Gamma} + \breve{\Gamma}^T F(t) X, \tag{2.42}
$$

where

$$
X = \begin{bmatrix} E_a & E_h & 0 & 0 & 0 & 0 \end{bmatrix}, \breve{\Gamma} = \begin{bmatrix} D^T P & 0 & 0 & 0 & (\bar{h} - \delta) D^T R_1 & \delta D^T R_2 \end{bmatrix}.
$$

Then, (2.37) can be written as

$$
\breve{\Sigma} + \varepsilon^{-1} \breve{\Gamma}^T \breve{\Gamma} < 0, \tag{2.43}
$$

where Σ is defined in (2.34). Taking Schur complement, one obtains (2.41). □

To inspect the strongness of the above robust stability criteria, some numerical examples are considered in the next section.

2.5.1 Numerical examples

Two numerical examples are presented in this section to validate the robust stability criteria derived in the previous section using proposed delay-decomposition technique.

Example 2.3. Consider system (2.31) with

$$
A = \begin{bmatrix} -0.5 & -2 \\ 1 & -1 \end{bmatrix}, A_h = \begin{bmatrix} -0.5 & -1 \\ 0 & 0.6 \end{bmatrix}, D_a = D_d = I, E_a = E_h = diag\{0.2, 0.2\}.
$$

Using Corollary 2.2, the delay bound $\bar{h}$ obtained for varying N is shown in Fig. 2.2. For this perturbed system, it can also be observed that the delay bound is maximum at $N = 2$. The reason is same as the case of linear time-delay system of the form (2.1). As the delay interval for both the integral inequalities in (2.17) is halved. The bounding gap reduces for both, leading to improved result. But with further increase in N, the gap in bounding the first integral term increases as that particular integral interval increases leading to decrease in $\bar{h}$ even though gap in the bounding of the second integral decreases. Therefore, one always gets maximum delay value at $N = 2$. The delay bound ($\bar{h}$) is found to be 0.9021 at $N = 2$. Numerical comparison with existing results are presented in Table 2.3.

Example 2.4. Consider another example of system (2.31) with

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, A_h = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, D_a = D_h = I, E_a = \begin{bmatrix} 1.6 & 0 \\ 0 & 0.05 \end{bmatrix}, E_h = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}.$$

The variation of $\bar{h}$ for different N obtained using Corollary 2.2 is shown in Fig. 2.2. It is reported in [126] that for this system considering a certain $F(= I)$, the analytical limit of $\bar{h}$ is 1.3771. The result obtained using Corollary 2.2 is 1.3594, which is quite close to this analytical limit. A comparison of the Corollary 2.2 results with the existing ones is made in Table 2.3.

It can be seen that the present result is quite less conservative than the existing ones for both the examples.

Table 2.3: Comparison of delay bound ($\bar{h}$) for Example 2.3 and 2.4

	Methods	$\bar{h}$
Example 2.3	[68]	0.3513
	[111]	0.5799
	[34]	0.6812
	[167]	0.8435
	[126]	0.8542
	Corollary 2.2	0.9021 (N=2)
Example 2.4	[175]	0.2412
	[68]	0.2412
	[111]	0.7059
	[34]	1.1490
	[167]	1.1490
	Corollary 2.2	1.3594 (N=2)

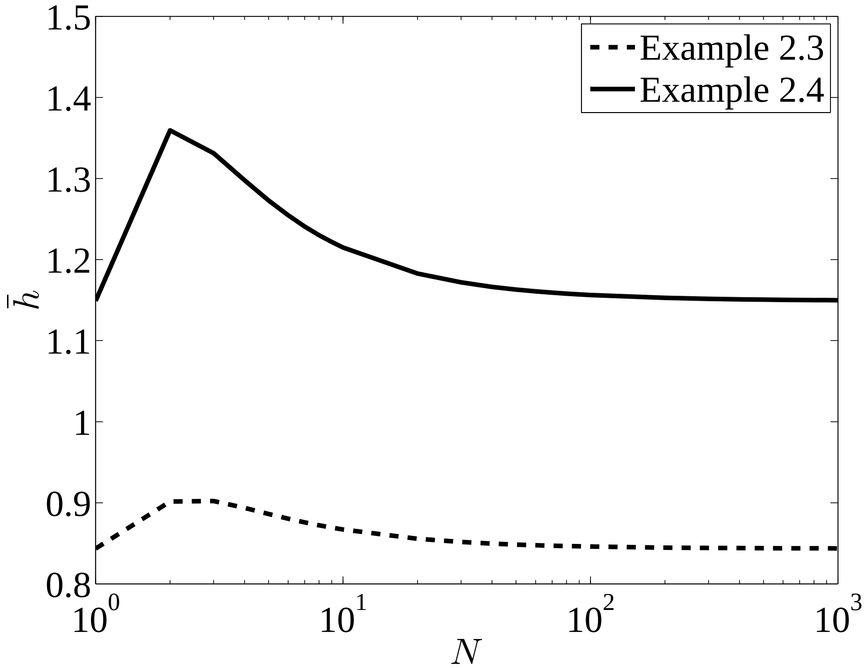

Figure 2.2: Variation of delay bound with no. of intervals for Example 2.3 and 2.4

2.6 Chapter summary

In this chapter, the following presentations have been made:

- A new less conservative criterion for stability analysis for systems with single delay has been proposed.
- The proposed delay-decomposition technique does not require a infinite-dimensional quadratic LK functional to derive the stability criterion.
- A sufficient stability criterion is derived by decomposing the whole delay range into several intervals and drawing a single one out of them by defining a simple multiple LK functional.
- The resulting criterion is independent of the number of decomposition of the delay interval as a result of which a finite-dimensional LMI is formulated.
- The complexity of the stability criterion does not increase with increase in number of delay decomposition.

- The stability analysis of the nominal system has been extended for robust analysis.
- Several numerical examples are presented to show the effectiveness of the proposed criterion, which show that the proposed criteria are less conservative than the existing ones while being computationally efficient for using lesser LMI variables.

CHAPTER 3

Stability analysis for systems with two delays

This chapter considers simple delay-dependent stability criterion for systems with two constant delays. Considering delays are small in nature, their ranges may overlap each other. Rather than treating the delays individually while defining Lyapunov-Krasovskii functionals, it may be advantageous if the overlapping feature of the delays can be exploited in the delay-dependent analysis for such systems. By extracting the overlapping feature of the two delays, a less conservative stability criterion is proposed in this chapter and the same is compared with the criterion derived by treating the delays individually. The same approach is used to derive a robust stability criterion for uncertain systems. Numerical examples are presented that validates the less-conservativeness of the proposed criteria.

3.1 Introduction

Often multiple delays are encountered in practical systems [120]. For example, a feedback control loop may introduce additional delays while another delay embedded into the plant model itself [78,179]. Multiple delay example is also found in networked control systems with both sensor to controller and controller to actuator delay [86,152,179]. The stability analysis of such systems with multiple delays is more complicated than that of systems with single delay. This motivates the stability analysis problem for systems affected by time delays. It may be noted that such multiple constant delays may have overlapping ranges due to their limited nature.

Existing attempts for stability analysis of systems with multiple time-delays [38, 78, 92, 179] are mere extensions of single delay approaches as the delays are treated individually in analysis [33, 36, 44]. Some stability criteria (delay-independent or delay-dependent) are proposed in [91] using LMI approach for system with multiple delays. The same LMI approach is used to derive less conservative robust stability condition in [122]. By making use of the characteristic equation of the system, [58] derives a delay-independent stability criterion in terms of the matrix measure and spectral norm of the matrix. To reduce the conservatism further, [58,166] propose a new delay-independent criterion in terms of the spectral radius of modulus matrices. To improve the conservatism, [178] uses a method to obtain new delay-independent stability criteria in terms of the spectral radius of modulus matrices based on characteristic equation of the system.

The robust stability analysis of time-delay systems with multiple delays is considerably investigated in literature. In order to obtain a less conservative criterion, a slack variable approach is used in [99]. A robust stability condition is derived in [13] in terms of LMIs by using a descriptor model transformation of the system and by applying Moon's inequality for bounding cross terms for a class of systems with input delay. In [55], a free matrix variable approach is used to obtain a less conservative robust stability result. Using the characteristic function of linear time-delay system with multiple delays, a stability criterion has been proposed to guarantee α-stability in [10]. In [122, 123], robust stability criteria for systems with multiple delays are derived in the form of LMIs.

To this end, it may be possible that suitable analysis by exploiting overlapping information of the delays may yield better results. Such an attempt is made in this chapter by considering delay-dependent stability analysis of systems with two constant delays.

In this chapter, two delay-dependent stability criteria have been proposed, the first of which is just an extension of simple delay-dependent analysis shown in §3.2.2 by treating the delays individually and the latter one is the analysis made by exploiting the overlapping information of the two delays. As the overlapping information is concerned, four different overlapping conditions do arise for defining the Lyapunov-Krasovskii functional. Considering such multiple functional for different overlapping situations, a single stability criterion is derived that satisfies stability requirement of all such situations. On consideration of numerical examples, it is observed that the approach exploiting the overlapping feature yields less conservative result compared to the individual treatment of the delays.

3.2 Stability analysis

This section investigates the stability analysis of linear systems with two delays. For the completeness of this work, this section includes stability analysis of linear systems with two delays by treating the delays individually. Then, a stability criterion is developed for the same system by exploiting the overlapping feature of the delays. To extract this feature a special type of simple LK functional is constructed.

3.2.1 System description and preliminaries

Consider a linear system with two constant delays

$$\dot{x}(t) = Ax(t) + A_1x(t-h_1) + A_2x(t-h_2), \tag{3.1}$$

where $x(t) \in \Re^n$ is the state; A, A_1 and A_2 are known real constant matrices; the time delays h_1 and h_2 are constant satisfying

$$0 \leq h_{m1} \leq h_1 \leq h_{M1}, 0 \leq h_{m2} \leq h_2 \leq h_{M2}. \tag{3.2}$$

To this end, as noted in the introduction section, the analysis for systems with single delay can easily be extended for systems with two delays. The following theorem presents such a result.

3.2.2 Stability criterion when delays treated individually

Theorem 3.1. *System (3.1) is stable if there exist $P > 0$, $Q_{ij} > 0$, $R_{ik} > 0$, $i = 1,2$ and $j = 1,2,\ldots,4$, and arbitrary matrices M_{ik}, N_{ik}, $k = 1,2$ satisfying these LMIs*

$$\begin{bmatrix} \Theta & \bar{h}_1\Phi_l & \bar{h}_2\Phi_m \\ * & -\bar{h}_1R_{12} & 0 \\ * & * & -\bar{h}_2R_{22} \end{bmatrix} < 0, \qquad l = 1,2 \quad \textit{and} \quad m = 3,4, \tag{3.3}$$

where

$$
\begin{aligned}
\Phi_1 &= \begin{bmatrix}0 & M_{11}^T & N_{11}^T & 0 & 0 & 0 & 0\end{bmatrix}^T, \Phi_2 = \begin{bmatrix}0 & 0 & M_{12}^T & N_{12}^T & 0 & 0 & 0\end{bmatrix}^T,\\
\Phi_3 &= \begin{bmatrix}0 & 0 & 0 & 0 & M_{21}^T & N_{21}^T & 0\end{bmatrix}^T, \Phi_4 = \begin{bmatrix}0 & 0 & 0 & 0 & 0 & M_{22}^T & N_{22}^T\end{bmatrix}^T,\\
\bar{h}_i &= (h_{Mi} - h_{mi}), \quad i = 1,2, \Theta = [\Theta_{ij}]_{i,j=1,\dots,7}, \quad with\\
\Theta_{11} &= PA + A^TP + \sum_{i=1}^{2}\sum_{j=1}^{3} Q_{ij} + A^T\Pi A - R_{11} - R_{21}, \Theta_{12} = R_{11}, \Theta_{13} = PA_1 + A^T\Pi A_1,\\
\Theta_{14} &= 0, \Theta_{15} = R_{21}, \Theta_{16} = PA_2 + A^T\Pi A_2, \Theta_{17} = 0, \Theta_{22} = -Q_{12} + Q_{14} - R_{11} + (M_{11} + M_{11}^T),\\
\Theta_{23} &= -M_{11} + N_{11}^T, \Theta_{24} = 0, \Theta_{25} = 0, \Theta_{26} = 0, \Theta_{27} = 0,\\
\Theta_{33} &= -Q_{13} - Q_{14} + A_1^T\Pi A_1 + (M_{12} + M_{12}^T) + (-N_{11} - N_{11}^T), \Theta_{34} = -M_{12} + N_{12}^T, \Theta_{35} = 0,\\
\Theta_{36} &= A_1^T\Pi A_2, \Theta_{37} = 0, \Theta_{44} = -Q_{11} + (-N_{12} - N_{12}^T), \Theta_{55} = -Q_{22} + Q_{24} - R_{21} + (M_{21} + M_{21}^T),\\
\Theta_{56} &= -M_{21} + N_{21}^T, \Theta_{57} = 0, \Theta_{66} = -Q_{23} - Q_{24} + A_2^T\Pi A_2 + (M_{22} + M_{22}^T) + (-N_{21} - N_{21}^T),\\
\Theta_{67} &= -M_{22} + N_{22}^T, \Theta_{77} = -Q_{21} + (-N_{22} - N_{22}^T), \Pi = \Pi_1 + \Pi_2, \Pi_1 = \left[h_{m1}^2 R_{11} + \bar{h}_1 R_{12}\right],\\
\Pi_2 &= \left[h_{m2}^2 R_{21} + \bar{h}_2 R_{22}\right].
\end{aligned}
$$

Proof. Consider a Lyapunov-Krasovskii functional as:

$$
\begin{aligned}
V(x_t, \dot{x}_t) = x^T(t)Px(t) + \sum_{i=1}^{2}\Bigg[\int_{t-h_{Mi}}^{t} x^T(\theta)Q_{i1}x(\theta)d\theta + \int_{t-h_{mi}}^{t} x^T(\theta)Q_{i2}x(\theta)d\theta\\
+ \int_{t-h_i}^{t} x^T(\theta)Q_{i3}x(\theta)d\theta + \int_{t-h_i}^{t-h_{mi}} x^T(\theta)Q_{i4}x(\theta)d\theta + h_{mi}\int_{t-h_{mi}}^{t}\int_{\theta}^{t} \dot{x}^T(\omega)R_{i1}\dot{x}(\omega)d\omega d\theta\\
+ \int_{t-h_{Mi}}^{t-h_{mi}}\int_{\theta}^{t} \dot{x}^T(\omega)R_{i2}\dot{x}(\omega)d\omega d\theta\Bigg].
\end{aligned}
\tag{3.4}
$$

Taking the time derivative of the energy functional along the trajectory of system (3.1) yields

$$
\begin{aligned}
\dot{V}(x_t, \dot{x}_t) = 2x^T(t)PAx(t) + 2x^T(t)PA_1x(t-h_1) + 2x^T(t)PA_2x(t-h_2) + \sum_{i=1}^{2}\\
\Bigg[\sum_{j=1}^{3} x^T(t)Q_{ij}x(t) - x^T(t-h_{Mi})Q_{i1}x(t-h_{Mi}) - x^T(t-h_{mi})\left(Q_{i2} - Q_{i4}\right)x(t-h_{mi})
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{j=3}^{4} x^T(t-h_i)Q_{ij}x(t-h_i) + \dot{x}^T(t)(h_{mi}^2 R_{i1} + \bar{h}_i R_{i2})\dot{x}(t) \\
&\left. -h_{mi}\int\limits_{t-h_{mi}}^{t} \dot{x}^T(\theta)R_{i1}\dot{x}(\theta)d\theta - \int\limits_{t-h_{Mi}}^{t-h_{mi}} \dot{x}^T(\theta)R_{i2}\dot{x}(\theta)d\theta \right].
\end{aligned}
\tag{3.5}
$$

Following Lemma 1.2, one may approximate the first integral term of (3.5) as

$$
-h_{mi}\int\limits_{t-h_{mi}}^{t} \dot{x}^T(\theta)R_{i1}\dot{x}(\theta)d\theta \le \begin{bmatrix} x(t) \\ x(t-h_{mi}) \end{bmatrix} \begin{bmatrix} -R_{i1} & R_{i1} \\ * & -R_{i1} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h_{mi}) \end{bmatrix}. \tag{3.6}
$$

The last integral term of (3.5) may be written as

$$
-\int\limits_{t-h_{Mi}}^{t-h_{mi}} \dot{x}^T(\theta)R_{i2}\dot{x}(\theta)d\theta = -\int\limits_{t-h_{Mi}}^{t-h_i} \dot{x}^T(\theta)R_{i2}\dot{x}(\theta)d\theta - \int\limits_{t-h_i}^{t-h_{mi}} \dot{x}^T(\theta)R_{i2}\dot{x}(\theta)d\theta. \tag{3.7}
$$

Following Lemma 1.2, the above terms may be approximated as

$$
\begin{gathered}
-\int\limits_{t-h_{Mi}}^{t-h_i} \dot{x}^T(\theta)R_{i2}\dot{x}(\theta)d\theta \le \\
\begin{bmatrix} x(t-h_i) \\ x(t-h_{Mi}) \end{bmatrix}^T \left\{ \begin{bmatrix} M_{i1}+M_{i1}^T & -M_{i1}+N_{i1}^T \\ * & -N_{i1}-N_{i1}^T \end{bmatrix} + \bar{h}_i\rho_i \begin{bmatrix} M_{i1} \\ N_{i1} \end{bmatrix} R_{i2}^{-1} \begin{bmatrix} M_{i1} \\ N_{i1} \end{bmatrix}^T \right\} \begin{bmatrix} x(t-h_i) \\ x(t-h_{Mi}) \end{bmatrix},
\end{gathered}
\tag{3.8}
$$

and

$$
\begin{gathered}
-\int\limits_{t-h_i}^{t-h_{mi}} \dot{x}^T(\theta)R_{i2}\dot{x}(\theta)d\theta \le \\
\begin{bmatrix} x(t-h_{mi}) \\ x(t-h_i) \end{bmatrix}^T \left\{ \begin{bmatrix} M_{i2}+M_{i2}^T & -M_{i2}+N_{i2}^T \\ * & -N_{i2}-N_{i2}^T \end{bmatrix} + \bar{h}_i(1-\rho_i) \begin{bmatrix} M_{i2} \\ N_{i2} \end{bmatrix} R_{i2}^{-1} \begin{bmatrix} M_{i2} \\ N_{i2} \end{bmatrix}^T \right\} \begin{bmatrix} x(t-h_{mi}) \\ x(t-h_i) \end{bmatrix},
\end{gathered}
\tag{3.9}
$$

where $\rho_i = (h_i - h_{mi})/\bar{h}_i$, $0 \le \rho_i \le 1$.

Substituting (3.6), (3.8) and (3.9) in (3.5), one may write

$$\begin{aligned}\dot{V}\left(x_t, \dot{x}_t\right) \leq \xi^T(t)\left\{\Theta + \bar{h}_1\rho_1\Phi_1 R_{12}^{-1}\Phi_1^T + \bar{h}_1(1-\rho_1)\Phi_2 R_{12}^{-1}\Phi_2^T\right. \\ \left.+\bar{h}_2\rho_2\Phi_3 R_{22}^{-1}\Phi_3^T + \bar{h}_2(1-\rho_2)\Phi_4 R_{22}^{-1}\Phi_4^T\right\}\xi(t),\end{aligned} \tag{3.10}$$

where

$$\xi(t) = \left[x^T(t) \quad x^T(t-h_{m1}) \quad x^T(t-h_1) \quad x^T(t-h_{M1}) \; x^T(t-h_{m2}) \quad x^T(t-h_2) \quad x^T(t-h_{M2})\right]^T.$$

The above equation (3.10) is polytope of matrices and is always negative definite if the following conditions are satisfied:

$$\Theta + \left(\bar{h}_1\Phi_l\right)\left\{\bar{h}_1 R_{12}\right\}^{-1}\left(\bar{h}_1\Phi_l\right)^T + \left(\bar{h}_2\Phi_m\right)\left\{\bar{h}_2 R_{22}\right\}^{-1}\left(\bar{h}_2\Phi_m\right)^T < 0, \tag{3.11}$$

where $l = 1, 2$ and $m = 3, 4$.

Applying Schur complement twice on (3.11), one obtains (3.3). Hence, the theorem is proved. □

This section provides us results on stability analysis of systems with two delays by treating the delays individually. Next, the stability analysis of (3.1) by extracting the overlapping feature of the delays is presented.

3.2.3 Stability criterion exploiting overlapping delay ranges

In this section, a delay-dependent criterion is developed by using the overlapping information of the delays. For systems with two delays, four different situations may arise based on the delay values taken in different sub-intervals generated from the overlapping feature of the two delay ranges as shown in Fig. 3.1. Particular delay values (h_1, h_2) lying in different sub-intervals are shown in the plot (identified by the dotted line) and the value of the delay ranges are identified at the bottom along the x-axis of the graph. These situations are further exploited to define a suitable Lyapunov-Krasovskii functional for obtaining a less conservative criterion. This result is presented next.

Theorem 3.2. *System (3.1) is stable if there exist $P > 0$, $Q_{ij} > 0$, $R_{ik} > 0$, $i = 1, 2$ and*

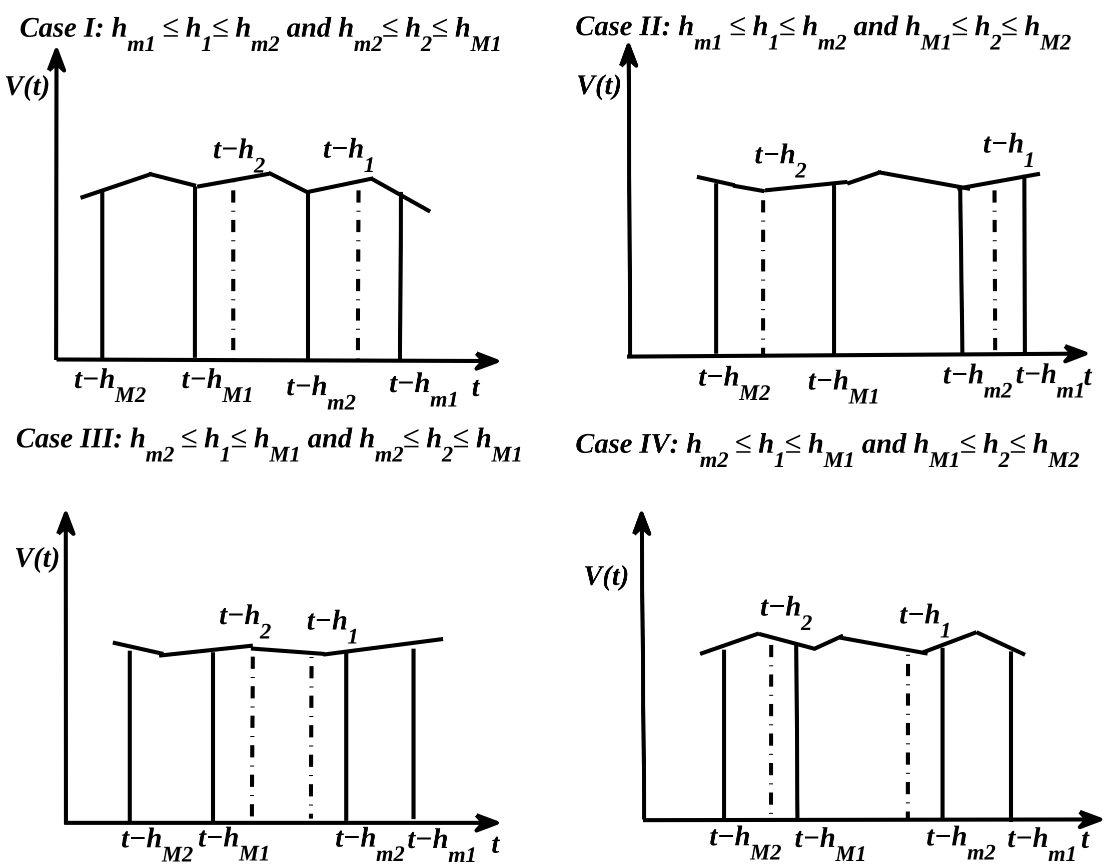

Figure 3.1: Situations arising out of overlapping nature of delays

$j = 1, 2, \ldots, 4$, and arbitrary matrices M_{jk}, N_{jk}, $k = 1, 2$ satisfying these LMIs:

$$\begin{bmatrix} \bar{\Theta} & \sigma_1\phi_l & \sigma_2\phi_m \\ * & -\sigma_1 R_{12} & 0 \\ * & * & -\sigma_2 R_{22} \end{bmatrix} < 0, \qquad \textit{where} \quad l = 1, 2 \quad \textit{and} \quad m = 3, 4, \tag{3.12}$$

where

$$\begin{aligned}
\phi_1 &= \begin{bmatrix} 0 & \alpha M_{11}^T & N_{11}^T & 0 & \bar{\alpha} M_{11}^T & 0 & 0 \end{bmatrix}^T, \phi_2 = \begin{bmatrix} 0 & 0 & M_{12}^T & \bar{\alpha} N_{12}^T & \alpha N_{12}^T & 0 & 0 \end{bmatrix}^T, \\
\phi_3 &= \begin{bmatrix} 0 & 0 & 0 & \bar{\beta} M_{21}^T & \beta M_{21}^T & N_{21}^T & 0 \end{bmatrix}^T, \phi_4 = \begin{bmatrix} 0 & 0 & 0 & \beta N_{22}^T & 0 & M_{22}^T & \bar{\beta} N_{22}^T \end{bmatrix}^T, \\
\sigma_1 &= \alpha(h_{m2} - h_{m1}) + \bar{\alpha}(h_{M1} - h_{m2}), \sigma_2 = \beta(h_{M1} - h_{m2}) + \bar{\beta}(h_{M2} - h_{M1}),
\end{aligned}$$

$$
\begin{aligned}
\bar{\Theta} &= [\bar{\Theta}_{ij}]_{i,j=1,\ldots,7}, \quad with \bar{\Theta}_{11} = PA + A^T P - R_{11} - R_{21} + \sum_{i=1}^{2}\sum_{j=1}^{3} Q_{ij} + A^T \Xi A, \\
\bar{\Theta}_{12} &= \alpha R_{11}, \bar{\Theta}_{13} = PA_1 + A^T \Xi A_1, \bar{\Theta}_{14} = \bar{\beta} R_{21}, \bar{\Theta}_{15} = \bar{\alpha} R_{11} + \beta R_{21}, \bar{\Theta}_{16} = PA_2 + A^T \Xi A_2, \\
\bar{\Theta}_{17} &= 0, \bar{\Theta}_{22} = -Q_{12} + \alpha\left[Q_{14} - R_{11} + \left(M_{11} + M_{11}^T\right)\right], \bar{\Theta}_{23} = \alpha\left(-M_{11} + N_{11}^T\right), \bar{\Theta}_{24} = 0, \\
\bar{\Theta}_{25} &= 0, \bar{\Theta}_{26} = 0, \bar{\Theta}_{27} = 0, \bar{\Theta}_{33} = -Q_{13} - Q_{14} + \left(-N_{11} - N_{11}^T\right) + \left(M_{12} + M_{12}^T\right) + A_1^T \Xi A_1, \\
\bar{\Theta}_{34} &= \bar{\alpha}\left(-M_{12} + N_{12}^T\right), \bar{\Theta}_{35} = \bar{\alpha}\left(-M_{11}^T + N_{11}\right) + \alpha\left(-M_{12} + N_{12}^T\right), \bar{\Theta}_{36} = A_1^T \Xi A_2, \\
\bar{\Theta}_{37} &= 0, \bar{\Theta}_{44} = -Q_{11} + \bar{\alpha}\left(-N_{12} - N_{12}^T\right) + \beta\left(-N_{22} - N_{22}^T\right) + \bar{\beta}\left[Q_{24} - R_{21} + \left(M_{21} + M_{21}^T\right)\right], \\
\bar{\Theta}_{45} &= 0, \bar{\Theta}_{46} = \beta\left(-M_{22}^T + N_{22}\right) + \bar{\beta}\left(-M_{21} + N_{21}^T\right), \bar{\Theta}_{47} = 0, \\
\bar{\Theta}_{55} &= -Q_{22} + \alpha\left(-N_{12} - N_{12}^T\right) + \bar{\alpha}\left[Q_{14} - R_{11} + \left(M_{11} + M_{11}^T\right)\right] + \beta\left[Q_{24} - R_{21} + \left(M_{21} + M_{21}^T\right)\right], \\
\bar{\Theta}_{56} &= \beta\left(-M_{21} + N_{21}^T\right), \bar{\Theta}_{57} = 0, \bar{\Theta}_{66} = -Q_{23} - Q_{24} + \left(-N_{21} - N_{21}^T\right) + \left(M_{22} + M_{22}^T\right) + A_2^T \Xi A_2, \\
\bar{\Theta}_{67} &= \bar{\beta}\left(-M_{22} + N_{22}^T\right), \bar{\Theta}_{77} = -Q_{21} + \bar{\beta}\left(-N_{22} - N_{22}^T\right), \Xi = \Xi_1 + \Xi_2, \\
\Xi_1 &= \alpha\left\{h_{m1}^2 R_{11} + (h_{m2} - h_{m1}) R_{12}\right\} + \bar{\alpha}\left\{h_{m2}^2 R_{11} + (h_{M1} - h_{m2}) R_{12}\right\}, \\
\Xi_2 &= \beta\left\{h_{m2}^2 R_{21} + (h_{M1} - h_{m2}) R_{22}\right\} + \bar{\beta}\left\{h_{M1}^2 R_{21} + (h_{M2} - h_{M1}) R_{22}\right\}, \\
\alpha, \beta &\in \begin{bmatrix} 0, & 1 \end{bmatrix}, \bar{\beta} = 1 - \beta, \bar{\alpha} = 1 - \alpha.
\end{aligned}
$$

Proof. To deal with the four cases presented in Fig. 3.1, consider a Lyapunov-Krasovskii functional by introducing two binary parameters $\alpha \in [0, 1]$ and $\beta \in [0, 1]$, different combinations of which represent the following cases.
Case I: $\alpha = 1$ and $\beta = 1$; Case II: $\alpha = 1$ and $\beta = 0$;
Case III: $\alpha = 0$ and $\beta = 1$; Case IV: $\alpha = 0$ and $\beta = 0$.

With the above introduction of α and β, a Lyapunov-Krasovskii functional is defined as:

$$
V\left(x_t, \dot{x}_t\right) = V_1(t) + \alpha V_2(t) + \bar{\alpha} V_3(t) + \beta V_4(t) + \bar{\beta} V_5(t), \tag{3.13}
$$

where

$$
\begin{aligned}
V_1(t) &= x^T(t) P x(t) + \sum_{i=1}^{2}\left[\int_{t-h_{Mi}}^{t} x^T(\theta) Q_{i1} x(\theta) d\theta + \int_{t-h_{mi}}^{t} x^T(\theta) Q_{i2} x(\theta) d\theta \right. \\
&\quad \left. + \int_{t-h_i}^{t} x^T(\theta) Q_{i3} x(\theta) d\theta \right],
\end{aligned}
$$

$$
\begin{aligned}
V_2(t) &= \int_{t-h_1}^{t-h_{m1}} x^T(\theta)Q_{14}x(\theta)d\theta + h_{m1}\int_{t-h_{m1}}^{t}\int_{\theta}^{t} \dot{x}^T(\omega)R_{11}\dot{x}(\omega)d\phi d\omega \\
&+ \int_{t-h_{m2}}^{t-h_{m1}}\int_{\theta}^{t} \dot{x}^T(\omega)R_{12}\dot{x}(\omega)d\omega d\theta, \\
V_3(t) &= \int_{t-h_1}^{t-h_{m2}} x^T(\theta)Q_{14}x(\theta)d\theta + h_{m2}\int_{t-h_{m2}}^{t}\int_{\theta}^{t} \dot{x}^T(\omega)R_{11}\dot{x}(\omega)d\phi d\theta \\
&+ \int_{t-h_{M1}}^{t-h_{m2}}\int_{\theta}^{t} \dot{x}^T(\omega)R_{12}\dot{x}(\omega)d\omega d\theta, \\
V_4(t) &= \int_{t-h_2}^{t-h_{m2}} x^T(\theta)Q_{24}x(\theta)d\theta + h_{m2}\int_{t-h_{m2}}^{t}\int_{\theta}^{t} \dot{x}^T(\omega)R_{21}\dot{x}(\omega)d\omega d\theta \\
&+ \int_{t-h_{M1}}^{t-h_{m2}}\int_{\theta}^{t} \dot{x}^T(\omega)R_{22}\dot{x}(\omega)d\phi d\theta, \\
V_5(t) &= \int_{t-h_2}^{t-h_{M1}} x^T(\theta)Q_{24}x(\theta)d\theta + h_{M1}\int_{t-h_{M1}}^{t}\int_{\theta}^{t} \dot{x}^T(\omega)R_{21}\dot{x}(\omega)d\omega d\theta \\
&+ \int_{t-h_{M2}}^{t-h_{M1}}\int_{\theta}^{t} \dot{x}^T(\omega)R_{22}\dot{x}(\omega)d\omega d\theta.
\end{aligned}
$$

The first component of LK functional ($V_1(t)$) is constituted of non-integral quadratic and integral quadratic terms corresponding to the two delays h_1 and h_2, which are similar to the terms considered in LK functional in Theorem 2.1. The term $V_1(t)$ is common to all the cases of overlapping delays. The other terms $V_2(t)$ to $V_5(t)$ are constituted with one single integral term and two double integral terms. The terms are written in such a way that corresponding to the similar terms in the functional considered in Theorem 2.1 but takes different ranges of the delays different α and β values are chosen.

Time-derivative of the energy functional (3.13) yields

$$
\dot{V}\left(x_t, \dot{x}_t\right) = \dot{V}_1(t) + \alpha\dot{V}_2(t) + \bar{\alpha}\dot{V}_3(t) + \beta\dot{V}_4(t) + \bar{\beta}\dot{V}_5(t), \tag{3.14}
$$

where

$$
\begin{aligned}
\dot{V}_1(t) &= 2x^T(t)P\dot{x}(t) + \sum_{i=1}^{2}\left[x^T(t)\left(Q_{i1}+Q_{i2}+Q_{i3}\right)x(t) - x^T(t-h_{Mi})Q_{i1}\right. \\
&\quad \left. x(t-h_{Mi}) - x^T(t-h_{mi})Q_{i2}x(t-h_{mi}) - x^T(t-h_i)\ Q_{i3}x(t-h_i)\right], \\
\dot{V}_2(t) &= x^T(t-h_{m1})Q_{14}x(t-h_{m1}) - x^T(t-h_1)Q_{14}x(t-h_1) \\
&\quad + \dot{x}^T(t)\left\{h_{m1}^2R_{11} + (h_{m2}-h_{m1})R_{12}\right\}\dot{x}(t) - h_{m1}\int\limits_{t-h_{m1}}^{t} x^T(\theta)R_{11}x(\theta)d\theta \\
&\quad - \int\limits_{t-h_{m2}}^{t-h_{m1}} \dot{x}^T(\theta)R_{12}\dot{x}(\theta)d\theta, \\
\dot{V}_3(t) &= x^T(t-h_{m2})Q_{14}x(t-h_{m2}) - x^T(t-h_1)Q_{14}x(t-h_1) \\
&\quad + \dot{x}^T(t)\left\{h_{m2}^2R_{11} + (h_{M1}-h_{m2})R_{12}\right\}\dot{x}(t) - h_{m2}\int\limits_{t-h_{m2}}^{t} x^T(\theta)R_{11}x(\theta)d\theta \\
&\quad - \int\limits_{t-h_{M1}}^{t-h_{m2}} \dot{x}^T(\theta)R_{12}\dot{x}(\theta)d\theta, \\
\dot{V}_4(t) &= x^T(t-h_{m2})Q_{24}x(t-h_{m2}) - x^T(t-h_2)Q_{24}x(t-h_2) \\
&\quad + \dot{x}^T(t)\left\{h_{m2}^2R_{21} + (h_{M1}-h_{m2})R_{22}\right\}\dot{x}(t) - h_{m2}\int\limits_{t-h_{m2}}^{t} \dot{x}^T(\theta)R_{21}\dot{x}(\theta)d\theta \\
&\quad - \int\limits_{t-h_{M1}}^{t-h_{m2}} \dot{x}^T(\theta)R_{22}\dot{x}(\theta)d\theta, \\
\dot{V}_5(t) &= x^T(t-h_{M1})Q_{24}x(t-h_{M1}) - x^T(t-h_2)Q_{24}x(t-h_2) \\
&\quad + \dot{x}^T(t)\left\{h_{M1}^2R_{21} + (h_{M2}-h_{M1})R_{22}\right\}\dot{x}(t) - h_{M1}\int\limits_{t-h_{M1}}^{t} \dot{x}^T(\theta)R_{21}\dot{x}(\theta)d\theta \\
&\quad - \int\limits_{t-h_{M2}}^{t-h_{M1}} \dot{x}^T(\theta)R_{22}\dot{x}(\theta)d\theta.
\end{aligned}
$$

Following lemma 1.2, one may approximate the integral terms as in Theorem 3.1 and then

(3.14) may be written as

$$\dot{V}(x_t, \dot{x}_t) \leq \xi^T(t)\left\{\bar{\Theta} + \sigma_1\rho_1\phi_1 R_{12}^{-1}\phi_1^T + \sigma_1(1-\rho_1)\phi_2 R_{12}^{-1}\phi_2^T \quad +\sigma_2\rho_2\phi_3 R_{22}^{-1}\phi_3^T \right. \\ \left. +\sigma_2(1-\rho_2)\phi_4 R_{22}^{-1}\phi_4^T\right\}\xi(t), \tag{3.15}$$

And (3.15) is polytope of matrices and is negative definite if it's two certain vertices are negative definite individually. Then, the stability requirement can be written as:

$$\bar{\Theta} + (\sigma_1\phi_l)\left\{\sigma_1 R_{12}\right\}^{-1}(\sigma_1\phi_l)^T + (\sigma_2\phi_m)\left\{\sigma_2 R_{22}\right\}^{-1}(\sigma_2\phi_m)^T < 0, \tag{3.16}$$

where $l = 1, 2$ and $m = 3, 4$.
Now, Taking Schur complement twice on (3.16), one obtains (3.12). Hence, the theorem is proved. $\square$

Remark 3.1. *It is well known that the LK approach followed in the chapter can exploit the lower bound information of the delay i.e., increase in lower bound parameter increases the tolerability in the upper bound value or the other way. By adopting the overlapping concept this lower bound or the upper bound for each of the intervals are changed so that the improvement happens.*

3.3 Numerical examples

In this section, two numerical examples are provided to demonstrate the effectiveness of the proposed criterion.

Example 3.1. Consider a time-delay system of the form [179]:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - h_1) + B_1 u(t) + B_2 u(t - h_2),$$

with

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ -0.5 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} -2 & -0.5 & 0 & 0 \\ -0.2 & -1 & 0 & 0 \\ 0.5 & 0 & -2 & -0.5 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix}^T, B_2 = \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix}^T.$$

Table 3.1: Comparison of delay bound ($\bar{h}_2$) for $\bar{h}_1 = 0.1$ for Example 3.1

Approach	$\bar{h}_1$	$\bar{h}_2$
Zhang, Wu, She and He [179]	0.1	0.56
Theorem 3.1	0.1	3.91
Theorem 3.2	0.1	3.95

In [179], for 0.1 value of $\bar{h}_1$, $\bar{h}_2$ is computed as 0.56 with a controller gain $K = [0.0129 \quad -0.0031 \quad -0.0009 \quad -0.3181]$. Using this controller (K) in the above system, one easily obtains the closed loop system of the form 3.1 with

$$A = \begin{bmatrix} 0.0129 & -0.0031 & -0.0009 & -0.3181 \\ 0.0129 & 0.4969 & -0.0009 & -0.3181 \\ -0.4871 & -0.0031 & 0.2991 & -0.3181 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} -2 & -0.5 & 0 & 0 \\ -0.2 & -1 & 0 & 0 \\ 0.5 & 0 & -2 & -0.5 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.0129 & -0.0031 & -0.0009 & -0.3181 \\ 0.0129 & -0.0031 & -0.0009 & -0.3181 \\ 0.0129 & -0.0031 & -0.0009 & -0.3181 \end{bmatrix}.$$

For this closed loop system, $\bar{h}_2$ is computed using Theorem 3.1 and Theorem 3.2. These values are tabulated in Table 3.1 along with the results obtained earlier in [179]. It can be seen that Theorem 3.2 is less conservative than that of the existing result [179] and Theorem 3.1. For further verification of less conservativeness, computed $\bar{h}_2$ using Theorem 3.1 and Theorem 3.2 are shown in Table 3.2 corresponding to different values of $\bar{h}_1$, which shows Theorem 3.2 is less conservative.

Example 3.2. Consider another example of (3.1) with

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ -2 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -1 \\ 0 & -3 \end{bmatrix}.$$

Using this Example, $\bar{h}_2$ is computed for given values of $\bar{h}_1$ and tabulated in the Table 3.3. The analytical value of the delay bound is computed using the direct method in [153] is

Table 3.2: Comparison of delay bound ($\bar{h}_2$) for different $\bar{h}_1$ for Example 3.1

$\bar{h}_1$	Theorem 3.1 ($\bar{h}_2$)	Theorem 3.2 ($\bar{h}_2$)
0.1	3.91	3.95
0.2	3.43	3.49
0.3	2.95	3.04
0.4	2.49	2.58
0.5	2.00	2.12

Table 3.3: Comparison of delay bound ($\bar{h}_2$) for Example 3.2

$\bar{h}_1$	Analytical Value ($\bar{h}_2$)	Theorem 3.1($\bar{h}_2$)		Theorem 3.2($\bar{h}_2$)	
0.10	1.414	0.93	65.77 %	0.95	67.18 %
0.15	1.206	0.84	69.65 %	0.86	71.31 %
0.20	1.154	0.75	64.99 %	0.77	66.72 %
0.25	1.120	0.63	56.25 %	0.65	58.03 %
0.30	1.024	0.51	49.80 %	0.53	51.75 %

shown in Table 3.3 for comparison. It is clearly seen from the table that Theorem 3.2 is less conservative in the sense of the determined maximum delay value while ensuring stability. Since delay value is positive scalar only, it is expressed in % compared to the analytical result in the table. The results obtained using Theorem 3.1 and Theorem 3.2 are also tabulated therein.

3.4 Robust stability analysis

In the previous section, the stability criteria for nominal systems with two delays are discussed. Since the proposed criterion is based on simple LK functional approach, one can easily use it for robust analysis of uncertain systems with two delays. The following section provides this result.

3.4.1 System description

Considering a class of uncertain linear time-delay system described by

$$\dot{x}(t) = \bar{A}x(t) + \bar{A}_1 x(t-h_1) + \bar{A}_2 x(t-h_2), \tag{3.17}$$

where $x(t) \in \Re^n$ is the state; the time delays h_1 and h_2 are continuous constant differentiable function satisfying

$$0 \leq h_{m1} \leq h_1 \leq h_{M1}, 0 \leq h_{m2} \leq h_2 \leq h_{M2}, \tag{3.18}$$

where $\bar{A} = (A + \Delta A(t))$, $\bar{A}_1 = (A_1 + \Delta A_1(t))$, $\bar{A}_2 = (A_2 + \Delta A_2(t))$; A, A_1 and A_2 are real-valued known constant matrices of appropriate dimensions, $\Delta A(t)$, $\Delta A_1(t)$ and $\Delta A_1(t)$ are real unknown matrices functions representing time-varying admissible norm-bounded uncertainties. These can be described as:

$$\Delta A(t) = D_0 F_0(t) E_0, \Delta A_1(t) = D_1 F_1(t) E_1, \Delta A_2(t) = D_2 F_2(t) E_2, \tag{3.19}$$

where $F_i(t)$, for $i = 0, 1, 2.$ are real unknown time-varying matrices with Lebesgue measurable elements satisfying

$$F_i^T(t) F_i(t) \leq I, \quad for \quad i = 0, 1, 2. \tag{3.20}$$

where D_0, D_1, D_2, E_0 , E_1 and E_2 are real known constant matrices.

In the next section, two delay-dependent robust criteria for system (3.17) have been included: one is by treating the delays individually and the other is by extracting the overlapping information of the delays.

3.4.2 Robust stability criterion when delays treated individually

The following robust stability criterion is developed for (3.17) by treating the delays individually.

Theorem 3.3. *System (3.17) is stable if there exist $P > 0$, $Q_{ij} > 0$, $R_{ik} > 0$, $i = 1, 2$, $j = 1, 2...4$ and arbitrary matrices M_{jk} ,N_{jk}, $k = 1, 2$ satisfying these LMIs:*

$$\begin{bmatrix} \Theta & \Sigma_0 & \Sigma_1 & \Sigma_2 \\ * & -\varepsilon_0 I & 0 & 0 \\ * & * & -\varepsilon_1 I & 0 \\ * & * & * & -\varepsilon_2 I \end{bmatrix} < 0, \tag{3.21}$$

where

$$
\begin{aligned}
\Theta &= \begin{bmatrix} \bar{\Theta} & \bar{h}_1\phi_l & \bar{h}_2\phi_m & \breve{A}^T\Xi \\ * & -\bar{h}_1R_{11} & 0 & 0 \\ * & * & -\bar{h}_2R_{22} & 0 \\ * & * & * & -\Xi \end{bmatrix}, for \quad l=1,2, \quad m=3,4, \\
\phi_1 &= \begin{bmatrix} 0 & M_{11}^T & N_{11}^T & 0 & 0 & 0 & 0 \end{bmatrix}^T, \phi_2 = \begin{bmatrix} 0 & 0 & M_{12}^T & N_{12}^T & 0 & 0 & 0 \end{bmatrix}^T, \\
\phi_3 &= \begin{bmatrix} 0 & 0 & 0 & 0 & M_{21}^T & N_{21}^T & 0 \end{bmatrix}^T, \phi_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & M_{22}^T & N_{22}^T \end{bmatrix}^T, \\
\bar{\Theta} &= [\bar{\Theta}_{ij}]_{i,j=1,..,7.}, \bar{\Theta}_{11} = PA + A^TP + \sum_{i=1}^{2}\sum_{j=1}^{3} Q_{ij} - R_{11} - R_{21} + \varepsilon_0 E_0^T E_0, \bar{\Theta}_{12} = R_{11}, \\
\bar{\Theta}_{13} &= PA_1, \bar{\Theta}_{14} = 0, \bar{\Theta}_{15} = R_{21}, \bar{\Theta}_{16} = PA_2, \bar{\Theta}_{17} = 0, \bar{\Theta}_{22} = -Q_{12} + Q_{14} - R_{11} + (M_{11} + M_{11}^T), \\
\bar{\Theta}_{23} &= -M_{11} + N_{11}^T, \bar{\Theta}_{24} = 0, \bar{\Theta}_{25} = 0, \bar{\Theta}_{26} = 0, \bar{\Theta}_{27} = 0, \\
\bar{\Theta}_{33} &= -(Q_{13} + Q_{14}) + (M_{12} + M_{12}^T) + (-N_{11} - N_{11}^T) + \varepsilon_1 E_1^T E_1, \bar{\Theta}_{34} = -M_{12} + N_{12}^T, \\
\bar{\Theta}_{35} &= 0, \bar{\Theta}_{36} = 0, \bar{\Theta}_{37} = 0, \bar{\Theta}_{44} = -Q_{11} + (-N_{12} - N_{12}^T), \bar{\Theta}_{45} = 0, \bar{\Theta}_{46} = 0, \bar{\Theta}_{47} = 0, \\
\bar{\Theta}_{55} &= -Q_{22} + Q_{24} - R_{21} + (M_{21} + M_{21}^T), \bar{\Theta}_{56} = -M_{21} + N_{21}^T, \bar{\Theta}_{57} = 0, \\
\bar{\Theta}_{66} &= -(Q_{23} + Q_{24}) + (M_{22} + M_{22}^T) + (-N_{21} - N_{21}^T) + \varepsilon_2 E_2^T E_2, \\
\bar{\Theta}_{67} &= -M_{22} + N_{22}^T, \bar{\Theta}_{77} = -Q_{21} + (-N_{22} - N_{22}^T), \Xi = \Xi_1 + \Xi_2, \Xi_1 = \left[h_{m1}^2 R_{11} + \bar{h}_1 R_{12}\right], \\
\Xi_2 &= \left[h_{m2}^2 R_{21} + \bar{h}_2 R_{22}\right], \bar{h}_i = (h_{Mi} - h_{mi}), i = 1,2. \breve{A} = \begin{bmatrix} A & 0 & A_1 & 0 & 0 & A_2 & 0 \end{bmatrix}, \\
\Sigma_0 &= \begin{bmatrix} D_0^T P & \mathbf{0}_{1\times 8} & D_0^T\Xi \end{bmatrix}^T, \Sigma_1 = \begin{bmatrix} D_1^T P & \mathbf{0}_{1\times 8} & D_1^T\Xi \end{bmatrix}^T, \Sigma_2 = \begin{bmatrix} D_2^T P & \mathbf{0}_{1\times 8} & D_2^T\Xi \end{bmatrix}^T.
\end{aligned}
$$

Proof. To prove this theorem, consider LK functional as (3.4) (same as case of Theorem 3.1). The derivative of the functional is obtained as (3.5). The integral terms in the derivative of the functional is approximated using Lemma 1.2. Then, one obtains

$$
\begin{aligned}
\dot{V}(t) \leq \xi^T(t) \Big\{ & \hat{\Theta} + \bar{h}_1\rho_1\phi_1 R_{12}^{-1}\phi_1^T + \bar{h}_1(1-\rho_1)\phi_2 R_{12}^{-1}\phi_2^T \\
& + \bar{h}_2\rho_2\phi_3 R_{22}^{-1}\phi_3^T + \bar{h}_2(1-\rho_2)\phi_4 R_{22}^{-1}\phi_4^T \Big\} \xi(t),
\end{aligned} \tag{3.22}
$$

The above equation (3.22) is polytope of matrices and is always negative definite, if it's two certain vertices are so. Then, the stability requirement become

$$
\begin{aligned}
& \hat{\Theta} + \left(\bar{h}_1\phi_l\right)\left\{\bar{h}_1R_{12}\right\}^{-1}\left(\bar{h}_1\phi_l\right)^T + \left(\bar{h}_2\phi_m\right)\left\{\bar{h}_2R_{22}\right\}^{-1}\left(\bar{h}_2\phi_m\right)^T < 0, \\
& \qquad for \quad l = 1,2, \quad m = 3,4,
\end{aligned} \tag{3.23}
$$

where

$$
\begin{aligned}
\hat{\Theta} &= [\hat{\Theta}_{ij}]_{i,j=1,..,7}, \hat{\Theta}_{11} = P\bar{A} + \bar{A}^T P + \sum_{i=1}^{2}\sum_{j=1}^{3} Q_{ij} + \bar{A}^T \Xi \bar{A} - R_{11} - R_{21}, \hat{\Theta}_{12} = R_{11}, \\
\hat{\Theta}_{13} &= P\bar{A}_1 + \bar{A}^T \Xi \bar{A}_1, \hat{\Theta}_{14} = 0, \hat{\Theta}_{15} = R_{21}, \hat{\Theta}_{16} = P\bar{A}_2 + \bar{A}^T \Xi \bar{A}_2, \hat{\Theta}_{17} = 0, \\
\hat{\Theta}_{22} &= -Q_{12} + Q_{14} - R_{11} + (M_{11} + M_{11}^T), \hat{\Theta}_{23} = -M_{11} + N_{11}^T, \hat{\Theta}_{24} = 0, \hat{\Theta}_{25} = 0, \\
\hat{\Theta}_{26} &= 0, \hat{\Theta}_{27} = 0, \hat{\Theta}_{33} = -(Q_{13} + Q_{14}) + \bar{A}_1^T \Xi \bar{A}_1 + (M_{12} + M_{12}^T) + (-N_{11} - N_{11}^T), \\
\hat{\Theta}_{34} &= -M_{12} + N_{12}^T, \hat{\Theta}_{35} = 0, \hat{\Theta}_{36} = \bar{A}_1^T \Xi \bar{A}_2, \hat{\Theta}_{37} = 0, \hat{\Theta}_{44} = -Q_{11} + (-N_{12} - N_{12}^T), \\
\hat{\Theta}_{45} &= 0, \hat{\Theta}_{46} = 0, \hat{\Theta}_{47} = 0, \hat{\Theta}_{55} = -Q_{22} + Q_{24} - R_{21} + (M_{21} + M_{21}^T), \hat{\Theta}_{56} = -M_{21} + N_{21}^T, \\
\hat{\Theta}_{57} &= 0, \hat{\Theta}_{66} = -(Q_{23} + Q_{24}) + \bar{A}_2^T \Xi \bar{A}_2 + (M_{22} + M_{22}^T) + (-N_{21} - N_{21}^T), \\
\hat{\Theta}_{67} &= -M_{22} + N_{22}^T, \hat{\Theta}_{77} = -Q_{21} + (-N_{22} - N_{22}^T).
\end{aligned}
$$

The above equation may be written as

$$
\begin{gathered}
\breve{\Theta} + \left(\bar{h}_1 \phi_l\right) \left\{\bar{h}_1 R_{12}\right\}^{-1} \left(\bar{h}_1 \phi_l^T\right) + \left(\bar{h}_2 \phi_m\right) \left\{\bar{h}_2 R_{22}\right\}^{-1} \left(\bar{h}_2 \phi_m^T\right) + \breve{\bar{A}}^T \Xi \breve{\bar{A}} < 0, \\
for \quad l = 1, 2, \quad m = 3, 4,
\end{gathered}
\tag{3.24}
$$

where

$$
\begin{aligned}
\breve{\Theta} &= [\breve{\Theta}_{ij}]_{i,j=1,..,7}, \breve{\Theta}_{11} = P\bar{A} + \bar{A}^T P + \sum_{i=1}^{2}\sum_{j=1}^{3} Q_{ij} - \sum_{i=1}^{2} R_{i1}, \breve{\Theta}_{12} = \hat{\Theta}_{12}, \breve{\Theta}_{13} = P\bar{A}_1, \\
\breve{\Theta}_{14} &= 0, \breve{\Theta}_{15} = \hat{\Theta}_{15}, \breve{\Theta}_{16} = P\bar{A}_2, \breve{\Theta}_{17} = 0, \breve{\Theta}_{22} = \hat{\Theta}_{22}, \breve{\Theta}_{23} = \hat{\Theta}_{23}, \breve{\Theta}_{24} = 0, \breve{\Theta}_{25} = 0, \\
\breve{\Theta}_{26} &= 0, \breve{\Theta}_{27} = 0, \breve{\Theta}_{33} = -Q_{13} - Q_{14} + (M_{12} + M_{12}^T) + (-N_{11} - N_{11}^T), \breve{\Theta}_{34} = \hat{\Theta}_{34}, \\
\breve{\Theta}_{35} &= 0, \breve{\Theta}_{36} = 0, \breve{\Theta}_{37} = 0, \breve{\Theta}_{44} = \hat{\Theta}_{44}, \breve{\Theta}_{45} = 0, \breve{\Theta}_{46} = 0, \breve{\Theta}_{47} = 0, \breve{\Theta}_{55} = \hat{\Theta}_{55}, \breve{\Theta}_{56} = \hat{\Theta}_{56}, \\
\breve{\Theta}_{57} &= 0, \breve{\Theta}_{66} = -Q_{23} - Q_{24} + (M_{22} + M_{22}^T) + (-N_{21} - N_{21}^T), \breve{\Theta}_{67} = \hat{\Theta}_{67}, \breve{\Theta}_{77} = \hat{\Theta}_{77}, \\
\breve{\bar{A}} &= \begin{bmatrix} \bar{A} & 0 & \bar{A}_1 & 0 & 0 & \bar{A}_2 & 0 \end{bmatrix}.
\end{aligned}
$$

Taking Schur complement on (3.24), one obtains

$$
\begin{bmatrix}
\breve{\Theta} & \bar{h}_1 \phi_l & \bar{h}_2 \phi_m & \breve{\bar{A}}^T \Xi \\
* & -\bar{h}_1 R_{11} & 0 & 0 \\
* & * & -\bar{h}_2 R_{22} & 0 \\
* & * & * & -\Xi
\end{bmatrix} < 0, \quad for \quad l = 1, 2, \quad m = 3, 4. \tag{3.25}
$$

Separating the uncertain terms from (3.25) and treating all ΔA, ΔA_1 and ΔA_2 terms in separate matrices, one may write (3.25) as:

$$\tilde{\Theta} + \sum_{j=1}^{3}(\bar{\Sigma}_j^T F_j{}^T \Sigma_j{}^T + \Sigma_j F_j \bar{\Sigma}_j) < 0, \tag{3.26}$$

where

$$\begin{aligned}
\tilde{\Theta} &= \begin{bmatrix} \grave{\Theta} & \bar{h}_1\phi_l & \bar{h}_2\phi_m & \breve{A}^T\Xi \\ * & -\bar{h}_1 R_{11} & 0 & 0 \\ * & * & -\bar{h}_2 R_{22} & 0 \\ * & * & * & -\Xi \end{bmatrix}, \bar{\Sigma}_0 = \begin{bmatrix} E_0 & \mathbf{0}_{1\times 9} \end{bmatrix}, \bar{\Sigma}_1 = \begin{bmatrix} \mathbf{0}_{1\times 2} & E_1 & \mathbf{0}_{1\times 7} \end{bmatrix}, \\
\bar{\Sigma}_2 &= \begin{bmatrix} \mathbf{0}_{1\times 5} & E_2 & \mathbf{0}_{1\times 4} \end{bmatrix}, \Sigma_0 = \begin{bmatrix} D_0^T P & \mathbf{0}_{1\times 8} & D_0^T\Xi \end{bmatrix}^T, \Sigma_1 = \begin{bmatrix} D_1^T P & \mathbf{0}_{1\times 8} & D_1^T\Xi \end{bmatrix}^T, \\
\Sigma_2 &= \begin{bmatrix} D_2^T P & \mathbf{0}_{1\times 8} & D_2^T\Xi \end{bmatrix}^T, \grave{\Theta} = [\grave{\Theta}_{ij}]_{i,j=1,..,7}, \grave{\Theta}_{11} = PA + A^T P + \sum_{i=1}^{2}\sum_{j=1}^{3} Q_{ij} - \sum_{i=1}^{2} R_{i1}, \\
\grave{\Theta}_{12} &= \hat{\Theta}_{12}, \grave{\Theta}_{13} = PA_1, \grave{\Theta}_{14} = 0, \grave{\Theta}_{15} = \hat{\Theta}_{15}, \grave{\Theta}_{16} = PA_2, \grave{\Theta}_{17} = 0, \grave{\Theta}_{22} = \hat{\Theta}_{22}, \grave{\Theta}_{23} = \hat{\Theta}_{23}, \\
\grave{\Theta}_{24} &= 0, \grave{\Theta}_{25} = 0, \grave{\Theta}_{26} = 0, \grave{\Theta}_{27} = 0, \grave{\Theta}_{33} = \breve{\Theta}_{33}, \grave{\Theta}_{34} = \hat{\Theta}_{34}, \grave{\Theta}_{35} = 0, \grave{\Theta}_{36} = 0, \grave{\Theta}_{37} = 0, \\
\grave{\Theta}_{44} &= \hat{\Theta}_{44}, \grave{\Theta}_{45} = 0, \grave{\Theta}_{46} = 0, \grave{\Theta}_{47} = 0, \grave{\Theta}_{55} = \hat{\Theta}_{55}, \grave{\Theta}_{56} = \hat{\Theta}_{56}, \grave{\Theta}_{57} = 0, \grave{\Theta}_{66} = \breve{\Theta}_{66}, \\
\grave{\Theta}_{67} &= \hat{\Theta}_{67}, \grave{\Theta}_{77} = \hat{\Theta}_{77}.
\end{aligned}$$

Now, using Lemma 2.2, one obtains

$$\tilde{\Theta} + \sum_{j=1}^{3}\varepsilon_j \bar{\Sigma}_j^T \bar{\Sigma}_j + \sum_{j=1}^{3}\varepsilon_j^{-1}\Sigma_j^T \Sigma_j < 0, \tag{3.27}$$

where $\tilde{\Theta}$ is defined in (3.26). Combining the first two terms of (3.27), one obtains

$$\Theta + \sum_{j=1}^{3}\varepsilon_j^{-1}\Sigma_j^T \Sigma_j < 0, \tag{3.28}$$

where Θ is defined in (3.21). Finally, taking Schur complement, one obtains (3.21). This proves the above theorem. □

3.4.3 Robust stability criterion using overlapping treatment

Similar to the previous analysis in §3.2.3 for nominal system by extracting the overlapping feature of the delays, the overlapping feature of the delays can be extracted to reduce conservatism in analysis for uncertain systems. The following robust stability criterion holds such a result.

Theorem 3.4. *System (3.17) is stable if there exist $P > 0$, $Q_{ij} > 0$, $R_{ik} > 0$, $i = 1, 2$, $j = 1, 2..4$ and arbitrary matrices M_{jk} ,N_{jk} $k = 1, 2$ satisfying these LMIs:*

$$\begin{bmatrix} \Delta & \Sigma_0 & \Sigma_1 & \Sigma_2 \\ * & -\varepsilon_0 I & 0 & 0 \\ * & * & -\varepsilon_1 I & 0 \\ * & * & * & -\varepsilon_2 I \end{bmatrix} < 0, \quad for \quad l = 1, 2, \quad m = 3, 4. \tag{3.29}$$

where

$$\begin{aligned}
\Delta &= \begin{bmatrix} \bar{\Delta} & \Omega_1 \Phi_l & \Omega_2 \Phi_m & \breve{A}^T \Pi \\ * & -\Omega_1 R_{11} & 0 & 0 \\ * & * & -\Omega_2 R_{22} & 0 \\ * & * & * & -\Pi \end{bmatrix}, \quad for \quad l = 1, 2, \quad m = 3, 4, \\
\bar{\Delta} &= [\bar{\Delta}_{ij}]_{i,j=1,..,7}, \bar{\Delta}_{11} = PA + A^T P - R_{11} - R_{21} + \sum_{i=1}^{2} \sum_{j=1}^{3} Q_{ij} + \varepsilon_0 E_0^T E_0, \\
\bar{\Delta}_{12} &= \alpha R_{11}, \bar{\Delta}_{13} = PA_1, \bar{\Delta}_{14} = \bar{\beta} R_{21}, \bar{\Delta}_{15} = \bar{\alpha} R_{11} + \beta R_{21}, \bar{\Delta}_{16} = PA_2, \bar{\Delta}_{17} = 0, \\
\bar{\Delta}_{22} &= -Q_{12} + \alpha \left[Q_{14} - R_{11} + \left(M_{11} + M_{11}^T \right) \right], \bar{\Delta}_{23} = \alpha \left(-M_{11} + N_{11}^T \right), \bar{\Delta}_{24} = 0, \\
\bar{\Delta}_{25} &= 0, \bar{\Delta}_{26} = 0, \bar{\Delta}_{27} = 0, \bar{\Delta}_{33} = -Q_{13} - Q_{14} + \left(-N_{11} - N_{11}^T \right) + \left(M_{12} + M_{12}^T \right) + \varepsilon_1 E_1^T E_1, \\
\bar{\Delta}_{34} &= \bar{\alpha} \left(-M_{12} + N_{12}^T \right), \bar{\Delta}_{35} = \bar{\alpha} \left(-M_{11}^T + N_{11} \right) + \alpha \left(-M_{12} + N_{12}^T \right), \bar{\Delta}_{36} = 0, \bar{\Delta}_{37} = 0, \\
\bar{\Delta}_{44} &= -Q_{11} + \bar{\alpha} \left(-N_{12} - N_{12}^T \right) + \beta \left(-N_{22} - N_{22}^T \right) + \bar{\beta} \left[Q_{24} - R_{21} + \left(M_{21} + M_{21}^T \right) \right], \\
\bar{\Delta}_{45} &= 0, \bar{\Delta}_{46} = \beta \left(-M_{22}^T + N_{22} \right) + \bar{\beta} \left(-M_{21} + N_{21}^T \right), \bar{\Delta}_{47} = 0, \\
\bar{\Delta}_{55} &= -Q_{22} + \alpha \left(-N_{12} - N_{12}^T \right) + \bar{\alpha} \left[Q_{14} - R_{11} + \left(M_{11} + M_{11}^T \right) \right] \\
&\quad + \beta \left[Q_{24} - R_{21} + \left(M_{21} + M_{21}^T \right) \right], \bar{\Delta}_{56} = \beta \left(-M_{21} + N_{21}^T \right), \\
\bar{\Delta}_{57} &= 0, \bar{\Delta}_{66} = -Q_{23} - Q_{24} + \left(-N_{21} - N_{21}^T \right) + \left(M_{22} + M_{22}^T \right) + \varepsilon_2 E_2^T E_2, \\
\bar{\Delta}_{67} &= \bar{\beta} \left(-M_{22} + N_{22}^T \right), \bar{\Delta}_{77} = -Q_{21} + \bar{\beta} \left(-N_{22} - N_{22}^T \right), \Pi = \Pi_1 + \Pi_2, \\
\Pi_1 &= \alpha \left\{ h_{m1}^2 R_{11} + (h_{m2} - h_{m1}) R_{12} \right\} + \bar{\alpha} \left\{ h_{m2}^2 R_{11} + (h_{M1} - h_{m2}) R_{12} \right\}, \\
\Pi_2 &= \beta \left\{ h_{m2}^2 R_{21} + (h_{M1} - h_{m2}) R_{22} \right\} + \bar{\beta} \left\{ h_{M1}^2 R_{21} + (h_{M2} - h_{M1}) R_{22} \right\}, \\
\Phi_1 &= \begin{bmatrix} 0 & \alpha M_{11}^T & N_{11}^T & 0 & \bar{\alpha} M_{11}^T & \mathbf{0}_{1\times 2} \end{bmatrix}^T, \Phi_2 = \begin{bmatrix} \mathbf{0}_{1\times 2} & M_{12}^T & \bar{\alpha} N_{12}^T & \alpha N_{12}^T & \mathbf{0}_{1\times 2} \end{bmatrix}^T, \\
\Phi_3 &= \begin{bmatrix} \mathbf{0}_{1\times 3} & \bar{\beta} M_{21}^T & \beta M_{21}^T & N_{21}^T & 0 \end{bmatrix}^T, \Phi_4 = \begin{bmatrix} \mathbf{0}_{1\times 3} & \beta N_{22}^T & 0 & M_{22}^T & \bar{\beta} N_{22}^T \end{bmatrix}^T,
\end{aligned}$$

$$\begin{aligned}\Omega_1 &= \alpha(h_{m2}-h_{m1})+\bar{\alpha}(h_{M1}-h_{m2}), \Omega_2=\beta(h_{M1}-h_{m2})+\bar{\beta}(h_{M2}-h_{M1}).\\ \bar{\alpha} &= (1-\bar{\alpha}), \bar{\beta}=(1-\beta).\end{aligned}$$

Proof. Consider the same LK functional as (3.13). Obtain the derivative (3.13) as (3.14). Then approximate the integral terms by following Lemma 1.2, one obtains

$$\begin{aligned}\dot{V}(x_t,\dot{x}_t) \le \xi^T(t)\Big\{\hat{\Delta}+\Omega_1\rho_1\Phi_1R_1^{-1}\Phi_1^T+\Omega_1(1-\rho_1)\Phi_2R_1^{-1}\Phi_2^T\\ +\Omega_2\rho_2\Phi_3R_2^{-1}\Phi_3^T+\Omega_2(1-\rho_2)\Phi_4R_2^{-1}\Phi_4^T\Big\}\,\xi(t),\end{aligned} \tag{3.30}$$

where

$$\begin{aligned}\hat{\Delta} &= [\hat{\Delta}_{ij}]_{i,j=1,..,7}, \hat{\Delta}_{11}=P\bar{A}+\bar{A}^TP-R_{11}-R_{21}+\sum_{i=1}^{2}\sum_{j=1}^{3}Q_{ij}+\bar{A}^T\Pi\bar{A}, \hat{\Delta}_{12}=\bar{\Delta}_{12},\\ \hat{\Delta}_{13} &= P\bar{A}_1+\bar{A}^T\Pi\bar{A}_1, \hat{\Delta}_{14}=\bar{\Delta}_{14}, \hat{\Delta}_{15}=\bar{\Delta}_{15}, \hat{\Delta}_{16}=P\bar{A}_2+\bar{A}^T\Pi\bar{A}_2, \hat{\Delta}_{17}=0,\\ \hat{\Delta}_{22} &= \bar{\Delta}_{22}, \hat{\Delta}_{23}=\bar{\Delta}_{23}, \hat{\Delta}_{24}=0, \hat{\Delta}_{25}=0, \hat{\Delta}_{26}=0, \hat{\Delta}_{27}=0,\\ \hat{\Delta}_{33} &= -Q_{13}-Q_{14}+\left(-N_{11}-N_{11}^T\right)+\left(M_{12}+M_{12}^T\right)+\bar{A}_1^T\Pi\bar{A}_1, \hat{\Delta}_{34}=\bar{\Delta}_{34},\\ \hat{\Delta}_{35} &= \bar{\Delta}_{35}, \hat{\Delta}_{36}=\bar{A}_1^T\Pi\bar{A}_2, \hat{\Delta}_{37}=0, \hat{\Delta}_{44}=\bar{\Delta}_{44}, \hat{\Delta}_{45}=0, \hat{\Delta}_{46}=\bar{\Delta}_{46}, \hat{\Delta}_{47}=0, \hat{\Delta}_{55}=\bar{\Delta}_{55},\\ \hat{\Delta}_{56} &= \bar{\Delta}_{56}, \hat{\Delta}_{57}=0, \hat{\Delta}_{66}=-Q_{23}-Q_{24}+\left(-N_{21}-N_{21}^T\right)+\left(M_{22}+M_{22}^T\right)+\bar{A}_2^T\Pi\bar{A}_2,\\ \hat{\Delta}_{67} &= \bar{\Delta}_{67}, \hat{\Delta}_{77}=\bar{\Delta}_{77}, \Omega_1=\alpha(h_{m2}-h_{m1})+\bar{\alpha}(h_{M1}-h_{m2}),\\ \Omega_2 &= \beta(h_{M1}-h_{m2})+\bar{\beta}(h_{M2}-h_{M1}).\end{aligned}$$

The above equation (3.30) is polytope of matrices and is always negative definite if it's two certain vertices are so. Then, the stability requirement become

$$\hat{\Delta}+(\Omega_1\Phi_l)\left\{\Omega_1R_{12}\right\}^{-1}(\Omega_1\Phi_l)^T+(\Omega_2\Phi_m)\left\{\Omega_2R_{22}\right\}^{-1}(\Omega_2\Phi_m)^T<0, \tag{3.31}$$

By following Theorem 3.3, the criterion (3.29) can be derived from (3.31). Hence, the Theorem 3.4 is proved. □

The derived criteria is now verified by numerical example in the next section.

3.5 Numerical examples

In this section, the below example is considered to demonstrate the effectiveness of the proposed approach for robust analysis.

Table 3.4: Comparison of Delay Bound ($\bar{h}_2$) for Example 3.3

$\bar{h}_1$	Analytical Value ($\bar{h}_2$)	Theorem 3.3($\bar{h}_2$)		Theorem 3.4($\bar{h}_2$)	
0.13	0.581	0.260	44.82 %	0.275	47.33 %
0.14	0.575	0.230	40 %	0.265	46.08 %
0.15	0.563	0.210	37.30 %	0.255	45.29 %
0.16	0.552	0.180	32.60 %	0.245	44.38 %

Example 3.3. Consider a system of (3.18) with

$$\begin{aligned} A &= \begin{bmatrix} 1 & -1 \\ 0 & -10 \end{bmatrix}, A_1 = \begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} -3 & -1 \\ 2 & -2 \end{bmatrix}, \\ D_0 &= D_1 = D_2 = 0.001I, E_0 = E_1 = E_2 = I. \end{aligned}$$

In this Example, fixing the $h_{m1} = 0.010$ and $h_{m2} = 0.015$ corresponding $\bar{h}_2$ is computed for given values of $\bar{h}_1$ and tabulated in Table 3.4. It can be seen that Theorem 3.4 is less conservative than Theorem 3.3 that treats the two delays individually.

3.6 Chapter summary

This section highlights the contributions made in this chapter.

- By exploiting the overlapping range information of delays, new stability and consequently robust stability analysis results are obtained. They are less conservative than the approaches treating the delays individually.
- The proposed overlapping approach uses less number of matrix variables as compared to the approach treating the delays individually.

CHAPTER 4

Stabilization of systems with state delay

This chapter proposes stabilization criterion for systems with single delay via memoryless (static) state feedback controller. To derive such criterion, the proposed decomposition approach in Chapter 2 is used. The proposed approach yields simple, computationally efficient, comparatively less conservative LMI condition. The same is also used for stabilizer design of uncertain system. For both the cases, numerical examples are presented to show the effectiveness of the proposed approach.

4.1 Introduction

Control design for time-delay system is a problem of interest as it is well known that the delays are the major causes of instability and poor performance of control systems [43, 46, 139]. The numerical decomposition technique proposed in [43, 51] using complete quadratic LK functional gives a necessary and sufficient condition for stability of system with single constant delay. But the same is difficult to extend for control design problems. So approaches using simple LK functional has received a lot of attention as they can easily be extended for control design problems. Using such simple LK functional, many literature are available on reducing the conservatism by adopting a model transformation and/or bounding techniques [44, 45, 110]. The model transformation approaches yield conservative results as they may add some poles in system dynamics which are not present in the actual system, known as additional dynamics [44, 45]. To overcome this problem, a descriptor model transformation approach along with a bounding method of [111] has been used in [27, 33]. In [179], an integral-inequality approach is proposed to obtain a delay-dependent stabilization criterion for linear time-delay systems. It incorporates Moon's inequality [111] and the Leibniz-Newton formula to yield an integral inequality for quadratic terms.

In this chapter, the proposed decomposition technique in Chapter 2 is used to design static state feedback controller using simple LK functional. Even Though this approach gives sufficient condition for stability, it yields simple LMI condition as a solution. This is an important feature of the approach which can easily be extended for control design. During static state feedback control design for systems with constant delay, direct use of stability criteria leads to non-linear terms in the stabilization condition due to the involvement unknown parameter K. To handle such non-linear terms, in general bi-linear matrix inequality (BMI) approaches may be used to get the solution [105, 106]. Since the solution for BMI problems are not global so far and LMIs are still attractive for computational reasons, approximated LMI approaches are still attractive [89, 90]. The latter approach is used in this chapter to linearize the non-linear terms evolved due to the involvement of unknown K by quadratic transformation technique. As the above approach uses simple LK functional, it can easily be extended for robust stabilization criterion and the same is obtained in this chapter.

4.2 Stabilization using delay-decomposition

This section emphasizes on deriving the stabilization criterion for linear systems with constant delay. Before presenting the main stabilizing criterion, the system description is shown below.

4.2.1 System description

Consider a linear time-delay system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + B_2 u(t), \tag{4.1}$$

where $x(t) \in \Re^n$ is the system state, $u(t) \in \Re^n$ is the control input. A_0, A_1 and B_2 are appropriate dimensional matrices, h is a constant delay satisfying $0 \leq h \leq \bar{h}$. Let us define $x_t = \{x(t) : \quad t \in [-\bar{h} \quad 0]\}$. The initial condition for system (4.1), x_0 is first order differentiably smooth so that $\dot{x}_0$ exists and continuous. The system (4.1) is considered to be fully controllable and the states are measurable for feedback.
The objective in this chapter is to design a static state feedback controller of the form

$$u(t) = Kx(t), \tag{4.2}$$

for (4.1) using delay-discretized method proposed in Chapter 2. Using (4.2) in (4.1), the closed-loop system becomes

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + B_2 K x(t), \tag{4.3}$$

For decomposition scheme, the tolerable delay range $\bar{h}$ divided into N number of δ intervals of equal measure so that one may define

$$h_i = \begin{cases} 0 & \text{for} \quad i = 0, \\ i\delta & \text{for} \quad i = 1, 2, \ldots, N-1, \\ \bar{h} & \text{for} \quad i = N. \end{cases} \tag{4.4}$$

The following section holds the static state feedback stabilization criterion for (4.1) using the background knowledge of decomposition scheme proposed in Chapter 2.

4.2.2 Stabilization criterion

Now, the following theorem presents an LMI based static state feedback controller design for system (4.3).

Theorem 4.1. *System (4.3) is stable if there exist matrices* $\bar{P} > 0$, $\bar{Q}_j > 0$, $j = 1, \ldots, 4$, $\bar{R}_i > 0$ *and arbitrary matrices* $\bar{S}_l$, $\bar{M}_i$, $\bar{N}_i$, $l = 1 \ldots 5$, $i = 1, 2$, *that satisfy the following LMI:*

$$\begin{bmatrix} \Theta & \delta\bar{\Phi}_j \\ * & -\bar{R}_2 \end{bmatrix} < 0, \quad j = 1, 2, \tag{4.5}$$

where

$$\begin{aligned}
\bar{\Phi}_1 &= \begin{bmatrix} 0 & \bar{M}_1^T & \bar{N}_1^T & 0 & 0 \end{bmatrix}^T, \bar{\Phi}_2 = \begin{bmatrix} 0 & 0 & \bar{M}_2^T & \bar{N}_2^T & 0 \end{bmatrix}^T, \Theta = [\Theta_{ij}]_{i,j=1,\ldots,5}, \\
\Theta_{11} &= A_0\bar{S}_1^T + \bar{S}_1A_0^T + \sum_{k=1}^{3}\bar{Q}_k + B_2\bar{Y} + \bar{Y}^TB_2^T - \bar{R}_1, \Theta_{12} = \lambda\bar{S}_1A_0^T + \bar{R}_1 + \lambda\bar{Y}^TB_2^T, \\
\Theta_{13} &= A_1\bar{S}_1^T + \beta\bar{S}_1A_0^T + \beta\bar{Y}^TB_2^T, \Theta_{14} = \gamma\bar{S}_1A_0^T + \gamma\bar{Y}^TB_2^T, \\
\Theta_{15} &= \bar{P} - \bar{S}_1^T + \alpha\bar{S}_1A_0^T + \alpha\bar{Y}^TB_2^T, \Theta_{22} = -\left(\bar{Q}_2 - \bar{Q}_4\right) - \bar{R}_1 + \delta\left[\bar{M}_1 + \bar{M}_1^T\right], \\
\Theta_{23} &= \lambda A_1\bar{S}_1^T + \delta\left[-\bar{M}_1 + \bar{N}_1^T\right], \Theta_{24} = 0, \\
\Theta_{25} &= -\lambda\bar{S}_1^T, \Theta_{33} = \beta A_1\bar{S}_1^T + \beta\bar{S}_1A_1^T - \sum_{k=3}^{4}\bar{Q}_k + \delta\left[\bar{M}_2 + \bar{M}_2^T\right] + \delta\left[-\bar{N}_1 - \bar{N}_1^T\right], \\
\Theta_{34} &= \gamma\bar{S}_1A_1^T + \delta\left[-\bar{M}_2 + \bar{N}_2^T\right], \Theta_{35} = -\beta\bar{S}_1^T + \alpha\bar{S}_1A_1^T, \Theta_{44} = -\bar{Q}_1 + \delta\left[-\bar{N}_2 - \bar{N}_2^T\right], \\
\Theta_{45} &= -\gamma\bar{S}_1^T, \Theta_{55} = -\alpha\bar{S}_1^T - \alpha\bar{S}_1 + h_{(i-1)}^2\bar{R}_1 + \delta^2\bar{R}_2, \bar{S}_1 = S_1^{-1}, \bar{P} = \bar{S}_1P\bar{S}_1^T, \\
\bar{M}_i &= \bar{S}_1M_i\bar{S}_1^T, \bar{N}_i = \bar{S}_1N_i\bar{S}_1^T, \quad i = 1, 2, \bar{Q}_j = \bar{S}_1Q_j\bar{S}_1^T, \quad j = 1, ..4, \bar{Y} = K\bar{S}_1^T.
\end{aligned}$$

Proof. Consider a simple LK functional for i^{th} interval that $h \in [h_{(i-1)}, h_i]$ as:

$$V_i(x_t, \dot{x}_t) = x^T(t)Px(t) + \sum_{j=1}^{2}\int_{t-h_{(i+1-j)}}^{t} x^T(\theta)Q_jx(\theta)d\theta + \int_{t-h}^{t} x^T(\theta)Q_3x(\theta)d\theta$$
$$+ \int_{t-h}^{t-h_{(i-1)}} x^T(\theta)Q_4x(\theta)d\theta + h_{(i-1)}\int_{t-h_{(i-1)}}^{t}\int_{\theta}^{t}\dot{x}^T(\phi)R_1\dot{x}(\phi)d\phi d\theta + \delta\int_{t-h_i}^{t-h_{(i-1)}}\int_{\theta}^{t}\dot{x}^T(\phi)R_2\dot{x}(\phi)d\phi d\theta. \tag{4.6}$$

Differentiating V_i with respect to time along the state trajectory of (4.3) yields

$$\begin{aligned}\dot{V}_i(x_t, \dot{x}_t) = {} & 2x^T(t)P\dot{x}(t) + \sum_{k=1}^{3} x^T(t)Q_k x(t) - x^T(t-h_{(i-1)})(Q_2 - Q_4)x(t-h_{(i-1)}) \\ & - \sum_{k=3}^{4} x^T(t-h)Q_k x(t-h) - x^T(t-h_i)Q_1 x(t-h_i) + \dot{x}^T(t)\left\{h_{(i-1)}^2 R_1 + \delta^2 R_2\right\}\dot{x}(t) \\ & - h_{(i-1)} \int_{t-h_{(i-1)}}^{t} \dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta - \delta \int_{t-h_i}^{t-h_{(i-1)}} \dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta. \end{aligned} \tag{4.7}$$

Note that (4.7) does not incorporate any state information from the system. The conventional way to incorporate the state information in $\dot{V}_i$ is by replacing the $\dot{x}(t)$ term directly from the state equation (4.3). However, such replacement in stabilization problem of time-delay system does not yield a convenient LMI form as stabilization criterion. Alternatively, one may convert the state equation (4.3)suitably into a quadratic form and thereby appending the same to $\dot{V}_i$ term so that the replacement of $\dot{x}$ can be avoided. Such an approach in stabilization of time-delay systems has been used in [22]. Here, we use a quadratic form introducing new free variables S_1 to S_5 in order to explore the interplay of the different variables in the state dynamics.

Instead of replacing $\dot{x}(t)$ by directly using (4.3), we consider the quadratic formulation of the system dynamics (4.3) as:

$$\begin{aligned} & 2\left\{x^T(t)S_1 + x^T(t-h_{i-1})S_2 + x^T(t-h)S_3 + x^T(t-h_i)S_4 + \dot{x}^T(t)S_5\right\} \\ & \quad \times \{-\dot{x}(t) + A_0x(t) + A_1x(t-h) + B_2Kx(t)\} = 0, \end{aligned} \tag{4.8}$$

where $S_k, k = 1, \ldots, 5$ are arbitrary matrices of appropriate dimensions.

$$\begin{aligned}\dot{V}_i(x_t, \dot{x}_t) \leq {} & 2\xi^T(t)\begin{bmatrix} S_1^T & S_2^T & S_3^T & S_4^T & S_5^T \end{bmatrix}^T \{-\dot{x}(t) + A_0x(t) + A_1x(t-h) + B_2Kx(t)\} \\ & + 2x^T(t)P\dot{x}(t) + \sum_{k=1}^{3} x^T(t)Q_k x(t) - x^T(t-h_{(i-1)})(Q_2 - Q_4)x(t-h_{(i-1)}) \\ & - \sum_{k=3}^{4} x^T(t-h)Q_k x(t-h) - x^T(t-h_i)Q_1 x(t-h_i) + \dot{x}^T(t)\left\{h_{(i-1)}^2 R_1 + \delta^2 R_2\right\}\dot{x}(t) \\ & - h_{(i-1)} \int_{t-h_{(i-1)}}^{t} \dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta - \delta \int_{t-h_i}^{t-h_{(i-1)}} \dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta, \end{aligned} \tag{4.9}$$

where

$$\xi(t) \;=\; \begin{bmatrix} x^T(t) & x^T(t-h_{(i-1)}) & x^T(t-h) & x^T(t-h_i) & \dot{x}^T(t) \end{bmatrix}^T.$$

Following Lemma 1.2, the first integral in (4.9) satisfies

$$-h_{(i-1)} \int\limits_{t-h_{(i-1)}}^{t} \dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta \leq \begin{bmatrix} x(t) \\ x(t-h_{(i-1)}) \end{bmatrix} \begin{bmatrix} -R_1 & R_1 \\ * & -R_1 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h_{(i-1)}) \end{bmatrix}. \tag{4.10}$$

Last term in (4.9) may be written as:

$$-\delta \int\limits_{t-h_i}^{t-h_{(i-1)}} \dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta = -\delta \int\limits_{t-h_i}^{t-h} \dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta - \delta \int\limits_{t-h}^{t-h_{(i-1)}} \dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta. \tag{4.11}$$

Now, one requires to suitably replace the above integral terms in (4.9). One may approximate the integral terms in (4.11) as:

$$\begin{aligned} -\int\limits_{t-h}^{t-h_{(i-1)}} \dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta \quad &\leq \begin{bmatrix} x(t-h_{(i-1)}) \\ x(t-h) \end{bmatrix}^T \left\{ \begin{bmatrix} M_1+M_1^T & -M_1+N_1^T \\ * & -N_1-N_1^T \end{bmatrix} \right. \\ &\left. + \rho \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} R_2^{-1} \begin{bmatrix} M_1 \\ N_1 \end{bmatrix}^T \right\} \begin{bmatrix} x(t-h_{(i-1)}) \\ x(t-h) \end{bmatrix}. \end{aligned} \tag{4.12}$$

$$\begin{aligned} -\int\limits_{t-h_i}^{t-h} \dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta \quad &\leq \begin{bmatrix} x(t-h) \\ x(t-h_i) \end{bmatrix}^T \left\{ \begin{bmatrix} M_2+M_2^T & -M_2+N_2^T \\ * & -N_2-N_2^T \end{bmatrix} \right. \\ &\left. + (1-\rho) \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} R_2^{-1} \begin{bmatrix} M_2 \\ N_2 \end{bmatrix}^T \right\} \begin{bmatrix} x(t-h) \\ x(t-h_i) \end{bmatrix}. \end{aligned} \tag{4.13}$$

where $\rho = \frac{h-h_{i-1}}{\delta}$, $\quad 0 \leq \rho \leq 1$. Substituting (4.10), (4.12) and (4.13) into (4.11) and then that into (4.9), one may write

$$\dot{V}_i(x_t, \dot{x}_t) \leq \xi^T(t)(\Psi + h_{(i-1)}^2\Omega_i + \rho\delta^2\Phi_1R_2^{-1}\Phi_1^T + (1-\rho)\delta^2\Phi_2R_2^{-1}\Phi_2^T)\xi(t), \tag{4.14}$$

where

$$
\begin{aligned}
\Psi &= [\Psi_{ij}]_{i,j=1,\ldots,5}, \Psi_{11} = S_1A_0 + A_0^TS_1^T + \sum_{k=1}^{3} Q_k + S_1B_2K + K^TB_2^TS_1^T - R_1, \\
\Psi_{12} &= A_0^TS_2^T + R_1 + K^TB_2^TS_2^T, \Psi_{13} = S_1A_1 + A_0^TS_3^T + K^TB_2^TS_3^T, \Psi_{14} = A_0^TS_4^T + K^TB_2^TS_4^T, \\
\Psi_{15} &= P - S_1 + A_0^TS_5^T + K^TB_2^TS_5^T, \Psi_{22} = -(Q_2 - Q_4) - R_1 + \delta\left[M_1 + M_1^T\right], \\
\Psi_{23} &= S_2A_1 + \delta\left[-M_1 + N_1^T\right], \Psi_{24} = 0, \Psi_{25} = -S_2, \\
\Psi_{33} &= S_3A_1 + A_1^TS_3^T - \sum_{k=3}^{4} Q_k + \delta\left[M_2 + M_2^T\right] + \delta\left[-N_1 - N_1^T\right], \\
\Psi_{34} &= A_1^TS_4^T + \delta\left[-M_2 + N_2^T\right], \Psi_{35} = -S_3 + A_1^TS_5^T, \Psi_{44} = -Q_1 + \delta\left[-N_2 - N_2^T\right], \\
\Psi_{45} &= -S_4, \Psi_{55} = \delta^2R_2 - S_5 - S_5^T, \Omega_i = \begin{bmatrix} 0_{4n\times 4n} & 0_{4n\times n} \\ 0_{n\times 4n} & R_1 \end{bmatrix}.
\end{aligned}
$$

and Φ_1, Φ_2 are as given in (4.5). Therefore, the stability requirement for the i^{th} interval is

$$\Psi + h_{(i-1)}^2\Omega_i + \rho\delta^2\Phi_1R_2^{-1}\Phi_1^T + (1-\rho)\delta^2\Phi_2R_2^{-1}\Phi_2^T < 0. \tag{4.15}$$

The above is a polytope of matrices on ρ and is always negative definite. Then, (4.15) can be equivalently written as:

$$\Psi + h_{(i-1)}^2\Omega_i + \delta^2\Phi_jR_2^{-1}\Phi_j^T < 0, \quad j = 1, 2. \tag{4.16}$$

To this end, note that, $\Omega_i \geq 0$ and the term $h_{(i-1)}^2\Omega_i$ is maximum when $h \in [h_{(N-1)}, \bar{h}]$, the N^{th} interval. Therefore, irrespective of h lies in any of the intervals, the following condition ensures stability of (4.1):

$$\Psi + h_{(N-1)}^2\Omega_N + \delta^2\Phi_jR_2^{-1}\Phi_j^T < 0, \quad j = 1, 2. \tag{4.17}$$

Taking Schur complement for the last term in (4.17), since the third term in (4.17) is positive definite, one can write

$$\begin{bmatrix} \bar{\Psi} & \delta\Phi_k \\ * & -R_2 \end{bmatrix} < 0, \quad k = 1, 2, \tag{4.18}$$

where $\bar{\Psi} = \Psi + h_{(i-1)}^2\Omega_i$. For linearization, considering S_2, S_3, S_4 and S_5 as: $S_2 = \lambda S_1$,

$S_3 = \beta S_1$, $S_4 = \gamma S_1$, $S_5 = \alpha S_1$, and then, pre- and post-multiplying L.H.S. of (4.18) by

$$diag\left\{S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \quad S_1^{-1}\right\}$$

and its transpose respectively, and subsequently adopting the change of variables

$$\begin{aligned}\bar{S}_1 &= S_1^{-1}, \bar{P} = \bar{S}_1 P \bar{S}_1^T, \bar{M}_i = \bar{S}_1 M_i \bar{S}_1^T, \bar{N}_i = \bar{S}_1 N_i \bar{S}_1^T, i = 1, 2, \bar{Q}_j = \bar{S}_1 Q_j \bar{S}_1^T, j = 1, ..4, \\ \bar{Y} &= K\bar{S}_1^T.\end{aligned}$$

one obtains (4.5). Hence, the theorem is proved □

The stabilization criterion developed in Theorem 4.1 may be conservatively simplified by eliminating the free variables and reducing the dimension of the LMI. The following corollary presents this result.

Corollary 4.1. *System (4.3) is stable if there exist matrices $\bar{P} > 0$, $\bar{Q}_j > 0$, $j = 1, \ldots, 4$, $\bar{R}_i > 0$ and arbitrary matrices $\bar{S}_l$, $l = 1 \ldots 5$, $i = 1, 2$, that satisfy the following LMI:*

$$\bar{\Theta} < 0, \tag{4.19}$$

where

$$\begin{aligned}\bar{\Theta} &= [\bar{\Theta}_{ij}]_{i,j=1,\ldots,5}, \bar{\Theta}_{11} = A_0\bar{S}_1^T + \bar{S}_1 A_0^T + \sum_{k=1}^{3}\bar{Q}_k + B_2\bar{Y} + \bar{Y}^T B_2^T - \bar{R}_1, \\ \bar{\Theta}_{12} &= \lambda\bar{S}_1 A_0^T + \bar{R}_1 + \lambda\bar{Y}^T B_2^T, \bar{\Theta}_{13} = A_1\bar{S}_1^T + \beta\bar{S}_1 A_0^T + \beta\bar{Y}^T B_2^T, \bar{\Theta}_{14} = \gamma\bar{S}_1 A_0^T + \gamma\bar{Y}^T B_2^T, \\ \bar{\Theta}_{15} &= \bar{P} - \bar{S}_1^T + \alpha\bar{S}_1 A_0^T + \alpha\bar{Y}^T B_2^T, \bar{\Theta}_{22} = -\left(\bar{Q}_2 - \bar{Q}_4\right) - \bar{R}_1 - \bar{R}_2, \bar{\Theta}_{23} = \lambda A_1\bar{S}_1^T + \bar{R}_2, \\ \bar{\Theta}_{24} &= 0, \bar{\Theta}_{25} = -\lambda\bar{S}_1^T, \bar{\Theta}_{33} = \beta A_1\bar{S}_1^T + \beta\bar{S}_1 A_1^T - \sum_{k=3}^{4}\bar{Q}_k - 2\bar{R}_2, \bar{\Theta}_{34} = \gamma\bar{S}_1 A_1^T + \bar{R}_2, \\ \bar{\Theta}_{35} &= -\beta\bar{S}_1^T + \alpha\bar{S}_1 A_1^T, \bar{\Theta}_{44} = -\bar{Q}_1 - \bar{R}_2, \bar{\Theta}_{45} = -\gamma\bar{S}_1^T, \bar{\Theta}_{55} = h_{(i-1)}^2\bar{R}_1 + \delta^2\bar{R}_2 - \alpha(\bar{S}_1 + \bar{S}_1^T),\end{aligned}$$

Proof. Since (4.17) is positive definite, one may reduce the stability condition in the form of a single matrix inequalities as:

$$\Psi + h_{(N-1)}^2\Omega_N + \delta^2\Phi_1 R_2^{-1}\Phi_1^T + \delta^2\Phi_2 R_2^{-1}\Phi_2^T < 0. \tag{4.20}$$

One may write (4.20) as:

$$\bar{\Psi} + \delta^2\Phi_1 R_2^{-1}\Phi_1^T + \delta^2\Phi_2 R_2^{-1}\Phi_2^T < 0, \tag{4.21}$$

where $\bar{\Psi} = \Psi + h^2_{(N-1)}\Omega_N$. Separating the M_1, N_1, M_2 and N_2 terms from $\bar{\Psi}$, one obtains

$$\Upsilon + (\delta\Phi_1)I_1^T + I_1(\delta\Phi_1)^T + (\delta\Phi_1)R_2^{-1}(\delta\Phi_1)^T + (\delta\Phi_2)I_2^T + I_2(\delta\Phi_2)^T + (\delta\Phi_2)R_2^{-1}(\delta\Phi_2)^T < 0, \tag{4.22}$$

where

$$\begin{aligned}
\Upsilon &= [\Upsilon_{ij}]_{i,j=1,\ldots,5}, \Upsilon_{11} = S_1A_0 + A_0^TS_1^T + \sum_{k=1}^{3} Q_k + S_1B_2K + K^TB_2^TS_1^T - R_1, \\
\Upsilon_{12} &= A_0^TS_2^T + R_1 + K^TB_2^TS_2^T, \Upsilon_{13} = S_1A_1 + A_0^TS_3^T + K^TB_2^TS_3^T, \Upsilon_{14} = A_0^TS_4^T + K^TB_2^TS_4^T, \\
\Upsilon_{15} &= P - S_1 + A_0^TS_5^T + K^TB_2^TS_5^T, \Upsilon_{22} = -(Q_2 - Q_4) - R_1, \Upsilon_{23} = S_2A_1, \Upsilon_{24} = 0, \\
\Upsilon_{25} &= -S_2, \Upsilon_{33} = S_3A_1 + A_1^TS_3^T - \sum_{k=3}^{4} Q_k, \Upsilon_{34} = A_1^TS_4^T, \Upsilon_{35} = -S_3 + A_1^TS_5^T, \Upsilon_{44} = -Q_1, \\
\Upsilon_{45} &= -S_4, \Upsilon_{55} = -S_5 - S_5^T + \left(\bar{h} - \delta\right)^2 R_1 + \delta^2 R_2, I_1 = \begin{bmatrix} 0 & I & -I & 0 & 0 \end{bmatrix}^T, \\
I_2 &= \begin{bmatrix} 0 & 0 & I & -I & 0 \end{bmatrix}^T.
\end{aligned}$$

One can write (4.22) as:

$$\Upsilon + \left(\delta\Phi_1 + I_1R_2\right) R_2^{-1} \left(\delta\Phi_1 + I_1R_2\right)^T - I_1R_2I_1^T + \left(\delta\Phi_2 + I_2R_2\right) R_2^{-1} \left(\delta\Phi_2 + I_2R_2\right)^T - I_2R_2I_2^T < 0. \tag{4.23}$$

Next, following Lemma 1.2 and substituting the free variables as $M_i = {M_i}^T = -N_i = -{N_i}^T = -\delta^{-1}R_2$, the above stability condition yields

$$\bar{\Upsilon} < 0, \tag{4.24}$$

where

$$\begin{aligned}
\bar{\Upsilon} &= [\bar{\Upsilon}_{ij}]_{i,j=1,\ldots,5}, \bar{\Upsilon}_{11} = \Upsilon_{11}, \bar{\Upsilon}_{12} = \Upsilon_{12}, \bar{\Upsilon}_{13} = \Upsilon_{13}, \bar{\Upsilon}_{14} = \Upsilon_{14}, \bar{\Upsilon}_{15} = \Upsilon_{15}, \\
\bar{\Upsilon}_{22} &= -(Q_2 - Q_4) - R_1 - R_2, \bar{\Upsilon}_{23} = S_2A_1 + R_2, \bar{\Upsilon}_{24} = \Upsilon_{24}, \bar{\Upsilon}_{25} = \Upsilon_{25}, \\
\bar{\Upsilon}_{33} &= S_3A_1 + A_1^TS_3^T - \sum_{k=3}^{4} Q_k - 2R_2, \bar{\Upsilon}_{34} = A_1^TS_4^T + R_2, \bar{\Upsilon}_{35} = \Upsilon_{35}, \bar{\Upsilon}_{44} = -Q_1 - R_2, \\
\bar{\Upsilon}_{45} &= \Upsilon_{45}, \bar{\Upsilon}_{55} = \Upsilon_{55}.
\end{aligned}$$

For linearization, considering S_2, S_3, S_4 and S_5 as: $S_2 = \lambda S_1$, $S_3 = \beta S_1$, $S_4 = \gamma S_1$, $S_5 = \alpha S_1$, and then, pre- and post-multiplying by $diag\left\{S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \quad S_1^{-1}\right\}$, and its transpose respectively, and subsequently adopting the change of variables

$$\begin{aligned}\bar{S}_1 &= S_1^{-1}, \bar{P} = \bar{S}_1 P \bar{S}_1^T, \bar{M}_i = \bar{S}_1 M_i \bar{S}_1^T, \bar{N}_i = \bar{S}_1 N_i \bar{S}_1^T, i = 1, 2, \bar{Q}_j = \bar{S}_1 Q_j \bar{S}_1^T, j = 1, ..4, \\ \bar{Y} &= K \bar{S}_1^T.\end{aligned}$$

One obtains (4.19). □

To validate the proposed criteria, numerical examples are presented in the next subsection.

4.2.3 Numerical examples

Example 4.1. Consider a system [126] of the form (4.1) with

$$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

To obtain the controller gain using Theorem 4.1 for the above system, one uses LMI toolbox to solve the LMI proposed in Theorem 4.1. The external variables (λ, β, γ and α) are tuned using *fminsearch* function of MATLAB in order to obtain maximum tolerable delay. Note that, the obtained delay bound is not optimal, so one may obtain several different controller gain matrix (K) for same delay bound since the stabilization criterion is an LMI one that represents a set of solution itself.

Following the previous analysis in Chapter 2, recollect that one always gets maximum delay value at $N = 2$. For controller synthesis, a static state feedback controller is designed by setting $N = 2$. A comparison of maximum tolerable delay bound obtained using Theorem 4.1 and Corollary 4.1 along with existing results for this system is presented in Table 4.1. From the comparison, it is clear that the proposed decomposition approach yields less conservative result than the existing ones. To verify the stabilizing ability of the designed controller using Corollary 4.1, simulation result of the closed loop system with initial condition $x(t) = [2, -2]$, $t \in [-20, 0]$ is shown in Fig. 4.1. From the simulation, it is seen that the states of the closed-loop system are regulated to zero as expected. It may be noted that the controller gain obtained using Theorem 4.1 and Corollary 4.1 are not optimal since a different solution of the same LMI may be achieved for a different setting of the external variables λ, β, γ and α.

Table 4.1: Comparison of delay bound ($\bar{h}$)

Methods	$\bar{h}$	Controller gain (K)
[90]	0.6779	[-0.1155 -1.9839]
[34]	1.51	[-58.31 -294.935]
[179]	6	[-70.18 -77.67]
[126]	8	[-65.4058 -76.7778]
Corollary 4.1	18	[-7424.7 -7661.9]
Theorem 4.1	20	[-2852.3 -2959.7]

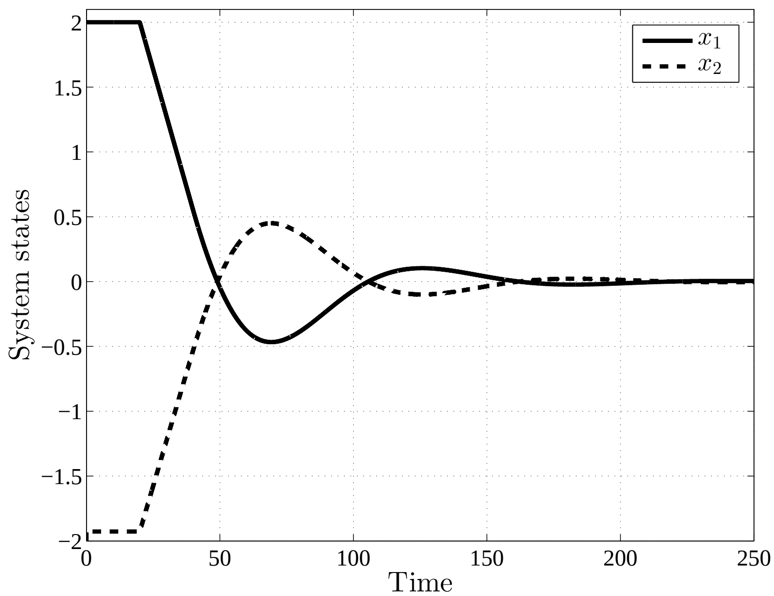

Figure 4.1: Variation of system states with respect to time for Example 4.1 using Theorem 4.1

Example 4.2. Let us consider a linearized model of a real-time aircraft control system [126], which is in the form of (4.1) with

$$A_0 = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.3681 & -0.7070 & 1.4200 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix}, A_1 = 0.3A_0, B_2 = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix}.$$

The state variables of this aircraft system is represented by $x = \begin{bmatrix} x_1^T & x_2^T & x_3^T & x_4^T \end{bmatrix}^T$, where

x_1 and x_2 are position (m) and velocity (m/s) of center of mass in spatial coordinates respectively, x_3 is rotation matrix (rad) of the body axes relative to the spatial axes and x_4 is body angular velocity vector (rad/s). For this case also, the controller is designed by setting $N = 2$. A comparison of maximum tolerable delay bounds using different approaches for this system is given in Table 4.2 below. From the comparison, it is clear that the proposed decomposition approach customarily gives less conservative result than the existing results. The obtained controller gain using Theorem 4.1 for maximum tolerable delay bound $(\bar{h} = 79)$ is used to simulate the closed loop system with initial condition $x(t) = [2, -4, 3, -5]$, $t \in [-79, 0]$. The variation of norm of the state vector with respect to time is shown in Fig. 4.2. From the simulation result, it is seen that the states of the closed loop system are stable. Note that, the plant states may not take such large values as in Fig. 4.2 in reality, rather it verifies mere stabilization performance carried out in this work. For actual system, the delay would be much lesser and one requires to include additional performance criteria for the design.

Table 4.2: Comparison of delay bound $(\bar{h})$

Methods	$\bar{h}$	Controller gain (K)
[90]	1.4142	$\begin{bmatrix} 13.6188 & 1.8680 & 0.7661 & -8.0951 \\ 21.9119 & 2.7268 & -0.1298 & -14.7952 \end{bmatrix}$
[126]	6	$\begin{bmatrix} -0.0458 & 0.1447 & 0.5490 & 0.2080 \\ -0.0187 & 0.1331 & 0.2516 & -0.4175 \end{bmatrix}$
Corollary 4.1	78	$\begin{bmatrix} -1.3518 & 0.2535 & 0.8412 & 2.1178 \\ -0.5552 & 0.2890 & 0.2580 & 0.4308 \end{bmatrix}$
Theorem 4.1	79	$\begin{bmatrix} 19.6354 & 2.4332 & 0.9949 & -10.9001 \\ 29.7580 & 3.4891 & 0.3834 & -18.1027 \end{bmatrix}$

4.3 Robust stabilization using delay-decomposition

The previous section covers the stabilizing control design technique for nominal system with single delay. But simple stabilization criterion does not work for systems with parametric uncertainty. For such systems, robust stabilization criterion works appropriately. The problem of robust stabilization is of recurring interest among control communities. To derive robust stabilization criterion for time-delay systems using Riccati-equation based method is

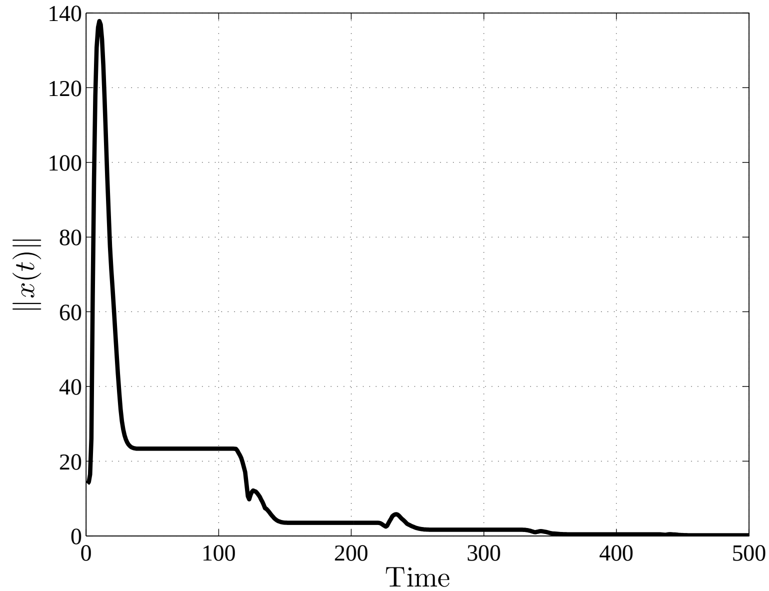

Figure 4.2: Variation of norm of the state vector with respect to time for Example 4.2 using Theorem 4.1

proposed in [102, 118, 119, 148]. In [12, 90], a linear matrix inequality approach is proposed for control design of systems with parametric uncertainty. The LMI approach has two advantages. First, it does not require tuning of parameters and/or matrix. Second, it can be efficiently solved numerically using interior-point algorithm. Robust stabilization using H_∞ control for uncertain system is investigated in [19]. A non-convex delay-dependent robust stabilization is obtained in [111] for uncertain system in the form of inequality to bind the cross-product terms. A descriptor model transformation approach is used in [32] to obtain robust stabilization criterion. In [179], an integral inequality approach is proposed to derive robust stabilization criterion to design memoryless controller. In [126], to derive stabilization criterion a quasi-full-size LK functional is chosen and free-weighted matrix approach is employed. In many literature, robust analysis approaches have been extended for robust stabilization [12,14,17,129]. But the decomposition approaches involving complete quadratic LK functional to obtain necessary and sufficient condition for robust stability are difficult to extend for robust stabilization.

In this section, the decomposition approach used for stabilization criterion is extended for robust stabilization criterion.

4.3.1 System description

Consider an uncertain system with unknown input delay

$$\dot{x}(t) = A_0(t)x(t) + A_1(t)x(t-h) + B_2(t)u(t), \tag{4.25}$$

where $A_0(t)$, $A_1(t)$ and $B_2(t)$ are the matrices with time-varying uncertainty and can be decomposed as:

$$A_0(t) = A_0 + \Delta A_0(t), \quad A_1(t) = A_1 + \Delta A_1(t), \quad B_2(t) = B_2 + \Delta B_2(t), \tag{4.26}$$

where $\Delta A_0(t)$, $\Delta A_1(t)$ and ΔB_2 are uncertain components of the nominal matrices A_0, A_1 and B_2 respectively. The uncertain matrices are norm bounded and can be decomposed as:

$$\begin{bmatrix} \Delta A_0(t) & \Delta A_1(t) & \Delta B_2(t) \end{bmatrix} = \begin{bmatrix} D_1F(t)E_1 & D_2F(t)E_2 & D_3F(t)E_3 \end{bmatrix}, \tag{4.27}$$

where D_1, D_2, D_3, E_1, E_2 and E_3 are appropriate dimensional constant matrices, and $F(t)$ satisfies $F^T(t)F(t) \le I$. h is constant delay. The objective of this section is to design a static state feedback controller $u(t) = Kx(t)$ as in (4.2) but for (4.25). Using (4.2) in (4.25), one obtains

$$\dot{x}(t) = A_0(t)x(t) + A_1(t)x(t-h) + B_2(t)Kx(t), \tag{4.28}$$

where K is the control gain to be designed so that the system can be stabilized. To design such controller, the following robust stabilization criterion is presented with the background knowledge of delay-decomposition approach adopted in §4.2.

4.3.2 Robust stabilization criterion

Now, the following theorem presents an LMI based static state feedback controller design for (4.25).

Theorem 4.2. *System (4.28) is stable if there exist matrices $\bar{P} > 0$, $\bar{Q}_j > 0$, $j = 1, \ldots, 4$, $\bar{R}_i > 0$ and arbitrary matrices $\bar{S}_l$, $\bar{M}_i$, $\bar{N}_i$, $l = 1 \ldots 5$, $i = 1, 2$, that satisfy the following LMI:*

$$\begin{bmatrix} \Pi_k & \check{E}_1 & \check{E}_2 & \check{E}_3 \\ * & -\varepsilon_1 I & 0 & 0 \\ * & * & -\varepsilon_2 I & 0 \\ * & * & * & -\varepsilon_3 I \end{bmatrix} < 0, k = 1, 2, \tag{4.29}$$

where

$$\begin{aligned}
\Pi_k &= \begin{bmatrix} \hat{\Pi} & \delta\bar{\phi}_l \\ * & -\bar{R}_2 \end{bmatrix}, \hat{\Pi} = \check{\Pi} + \bar{D}, \bar{D} = \sum_{k=1}^{3} \varepsilon_k \bar{D}_k^T \bar{D}_k, \bar{D}_k = \begin{bmatrix} D_k^T & \lambda D_k^T & \beta D_k^T & \gamma D_k^T & \alpha D_k^T \end{bmatrix}, \\
\bar{\phi}_1 &= \begin{bmatrix} 0 & \bar{M}_1^T & \bar{N}_1^T & \boldsymbol{0}_{1\times 2} \end{bmatrix}^T, \bar{\phi}_2 = \begin{bmatrix} \boldsymbol{0}_{1\times 2} & \bar{M}_2^T & \bar{N}_2^T & 0 \end{bmatrix}^T, \delta \triangleq \frac{\bar{h}}{N}, \text{ } N \textit{ is a positive integer}, \\
\check{\Pi} &= [\check{\Pi}_{ij}]_{i,j=1,..,5} \quad \textit{with}, \quad \check{\Pi}_{11} = \sum_{i=1}^{3} \bar{Q}_i - \bar{R}_1 + A_0 \bar{S}_1^T + \bar{S}_1 A_0^T + B_2 \bar{Y} + \bar{Y}^T B_2^T, \\
\check{\Pi}_{12} &= \bar{R}_1 + \lambda \bar{S}_1 A_0^T + \lambda \bar{Y}^T B_2^T, \check{\Pi}_{13} = \beta \bar{S}_1 A_0^T + \beta \bar{Y}^T B_2^T + A_1 \bar{S}_1^T, \\
\check{\Pi}_{14} &= \gamma \bar{S}_1 A_0^T + \gamma \bar{Y}^T B_2^T, \check{\Pi}_{15} = \bar{P} - \bar{S}_1^T + \alpha \bar{S}_1 A_0^T + \alpha \bar{Y}^T B_2^T, \\
\check{\Pi}_{22} &= \bar{Q}_4 - \bar{Q}_2 - \bar{R}_1 + \delta(\bar{M}_1 + \bar{M}_1^T), \check{\Pi}_{23} = \delta(-\bar{M}_1 + \bar{N}_1^T) + \lambda A_1 \bar{S}_1^T, \check{\Pi}_{24} = 0, \\
\check{\Pi}_{25} &= -\lambda \bar{S}_1^T, \check{\Pi}_{33} = -(\bar{Q}_3 + \bar{Q}_4) + \delta(-\bar{N}_1 - \bar{N}_1^T) + \delta(\bar{M}_2 + \bar{M}_2^T) + \beta A_1 \bar{S}_1^T + \beta \bar{S}_1 A_1^T \\
\check{\Pi}_{34} &= \gamma \bar{S}_1 A_1^T + \delta(-\bar{M}_2 + \bar{N}_2^T), \check{\Pi}_{35} = -\beta \bar{S}_1^T + \alpha \bar{S}_1 A_1^T, \check{\Pi}_{44} = -\bar{Q}_1 + \delta(-\bar{N}_2 - \bar{N}_2^T), \\
\check{\Pi}_{45} &= -\gamma \bar{S}_1^T, \check{\Pi}_{55} = -\alpha \bar{S}_1 - \alpha \bar{S}_1^T + \left\{ \left(h_{(i-1)}\right)^2 \bar{R}_1 + \delta^2 \bar{R}_2 \right\}, \check{E}_1 = \begin{bmatrix} E_1 \bar{S}_1^T & \boldsymbol{0}_{1\times 6} \end{bmatrix}^T, \\
\check{E}_2 &= \begin{bmatrix} 0 & 0 & E_2 \bar{S}_1^T & \boldsymbol{0}_{1\times 4} \end{bmatrix}^T, \check{E}_3 = \begin{bmatrix} E_3 \bar{Y} & \boldsymbol{0}_{1\times 6} \end{bmatrix}^T, \bar{S}_1 = S_1^{-1}, \bar{P} = \bar{S}_1 P \bar{S}_1^T, \\
\bar{M}_i &= \bar{S}_1 M_i \bar{S}_1^T, \bar{N}_i = \bar{S}_1 N_i \bar{S}_1^T, i = 1, 2, \bar{Q}_j = \bar{S}_1 Q_j \bar{S}_1^T, j = 1, ..4, \bar{Y} = K \bar{S}_1^T.
\end{aligned}$$

Proof. Consider a simple LK functional for i^{th} interval that $h \in [h_{(i-1)}, h_i]$

$$\begin{aligned}
V_i(x_t, \dot{x}_t) &= x^T(t) P x(t) + \sum_{j=1}^{2} \int_{t-h_{(i+1-j)}}^{t} x^T(\theta) Q_j x(\theta) d\theta + \int_{t-h}^{t} x^T(\theta) Q_3 x(\theta) d\theta \\
&+ \int_{t-h}^{t-h_{(i-1)}} x^T(\theta) Q_4 x(\theta) d\theta + h_{(i-1)} \int_{t-h_{(i-1)}}^{t} \int_{\theta}^{t} \dot{x}^T(\phi) R_1 \dot{x}(\phi) d\phi d\theta + \delta \int_{t-h_i}^{t-h_{(i-1)}} \int_{\theta}^{t} \dot{x}^T(\phi) R_2 \dot{x}(\phi) d\phi d\theta.
\end{aligned} \tag{4.30}$$

Differentiating V_i with respect to time along the state trajectory of (4.28) yields

$$\begin{aligned}
\dot{V}_i(x_t, \dot{x}_t) &= 2x^T(t) P \dot{x}(t) + \sum_{k=1}^{3} x^T(t) Q_k x(t) - x^T(t - h_{(i-1)})(Q_2 - Q_4) x(t - h_{(i-1)}) \\
&- \sum_{k=3}^{4} x^T(t-h) Q_k x(t-h) - x^T(t-h_i) Q_1 x(t-h_i) + \dot{x}^T(t) \left\{ h_{(i-1)}^2 R_1 + \delta^2 R_2 \right\} \dot{x}(t) \\
&- h_{(i-1)} \int_{t-h_{(i-1)}}^{t} \dot{x}^T(\theta) R_1 \dot{x}(\theta) d\theta - \delta \int_{t-h_i}^{t-h_{(i-1)}} \dot{x}^T(\theta) R_2 \dot{x}(\theta) d\theta.
\end{aligned} \tag{4.31}$$

Instead of replacing $\dot{x}(t)$ by directly using (4.28), we consider the quadratic formulation of the system dynamics (4.28) as:

$$\begin{aligned}&2\left\{x^T(t)S_1 + x^T(t-h_{i-1})S_2 + x^T(t-h)S_3 + x^T(t-h_i)S_4 + \dot{x}^T(t)S_5\right\}\\&\quad\times\{-\dot{x}(t) + A_0(t)x(t) + A_1(t)x(t-h) + B_2(t)Kx(t)\} = 0,\end{aligned} \tag{4.32}$$

where $S_k, k = 1,\ldots,5$ are arbitrary matrices of appropriate dimensions. Following (4.26), (4.27) and Lemma 2.2, one may write

$$\begin{aligned}&2\xi^T(t)\begin{bmatrix}S_1^T & S_2^T & S_3^T & S_4^T & S_5^T\end{bmatrix}^T\begin{bmatrix}D_1F(t)E_1x(t) & D_2F(t)E_2x(t-h) & D_3F(t)E_3Kx(t)\end{bmatrix}\\&\quad\le\sum_{k=1}^{3}\varepsilon_k\xi^T(t)\hat{D}_k^T\hat{D}_k\xi(t) + \varepsilon_1^{-1}x^T(t)E_1^TE_1x(t) + \varepsilon_2^{-1}x^T(t-h)E_2^TE_2x(t-h)\\&\qquad\qquad+\varepsilon_3^{-1}x^T(t)K^TE_3^TE_3Kx(t),\end{aligned} \tag{4.33}$$

where

$$\begin{aligned}\xi(t) &= \begin{bmatrix}x^T(t) & x^T(t-h_{(i-1)}) & x^T(t-h) & x^T(t-h_i) & \dot{x}^T(t)\end{bmatrix}^T,\\ \hat{D}_k &= \begin{bmatrix}D_k^TS_1^T & D_k^TS_2^T & D_k^TS_3^T & D_k^TS_4^T & D_k^TS_5^T\end{bmatrix}.\end{aligned}$$

Adding (4.32) to (4.31) by replacing the uncertain terms using (4.33), one obtains

$$\begin{aligned}\dot{V}_i(t) \le\; & 2\xi^T(t)\begin{bmatrix}S_1^T & S_2^T & S_3^T & S_4^T & S_5^T\end{bmatrix}^T\{-\dot{x}(t) + (A_0+B_2K)x(t) + A_1x(t-h)\}\\&+\sum_{k=1}^{3}\varepsilon_k\xi^T(t)\hat{D}_k^T\hat{D}_k\xi(t) + \varepsilon_1^{-1}x^T(t)E_1^TE_1x(t) + \varepsilon_2^{-1}x^T(t-h)E_2^TE_2x(t-h)\\&+\varepsilon_3^{-1}x^T(t)K^TE_3^TE_3Kx(t) + 2x^T(t)P\dot{x}(t) + \sum_{k=1}^{3}x^T(t)Q_kx(t)\\&-x^T(t-h_{(i-1)})(Q_2-Q_4)x(t-h_{(i-1)}) - \sum_{k=3}^{4}x^T(t-h)Q_kx(t-h)\\&-x^T(t-h_i)Q_1x(t-h_i) + \dot{x}^T(t)\left\{h_{(i-1)}^2R_1 + \delta^2R_2\right\}\dot{x}(t)\\&-h_{(i-1)}\int_{t-h_{(i-1)}}^{t}\dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta - \delta\int_{t-h_i}^{t-h_{(i-1)}}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta.\end{aligned} \tag{4.34}$$

Following Lemma 1.2, the first integral in (4.34) satisfies

$$-h_{(i-1)}\int_{t-h_{(i-1)}}^{t}\dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta \leq \begin{bmatrix} x(t) \\ x(t-h_{(i-1)}) \end{bmatrix}\begin{bmatrix} -R_1 & R_1 \\ * & -R_1 \end{bmatrix}\begin{bmatrix} x(t) \\ x(t-h_{(i-1)}) \end{bmatrix}. \qquad (4.35)$$

and the second one satisfies

$$\begin{aligned}
&-\delta\int_{t-h_{(i)}}^{t-h_{(i-1)}}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta \\
&= \begin{bmatrix} x(t-h_{(i-1)}) \\ x(t-h) \end{bmatrix}^T\left\{\begin{bmatrix} M_1+M_1^T & -M_1+N_1^T \\ * & -N_1-N_1^T \end{bmatrix}+\rho\begin{bmatrix} M_1 \\ N_1 \end{bmatrix}R_2^{-1}\begin{bmatrix} M_1 \\ N_1 \end{bmatrix}^T\right\}\begin{bmatrix} x(t-h_{(i-1)}) \\ x(t-h) \end{bmatrix} \\
&+\begin{bmatrix} x(t-h) \\ x(t-h_i) \end{bmatrix}^T\left\{\begin{bmatrix} M_2+M_2^T & -M_2+N_2^T \\ * & -N_2-N_2^T \end{bmatrix}+(1-\rho)\begin{bmatrix} M_2 \\ N_2 \end{bmatrix}R_2^{-1}\begin{bmatrix} M_2 \\ N_2 \end{bmatrix}^T\right\}\begin{bmatrix} x(t-h) \\ x(t-h_i) \end{bmatrix}.
\end{aligned} \qquad (4.36)$$

Then following the procedure from Theorem 4.1, one obtains

$$\dot{V}_i(t) \leq \xi^T(t)(\psi + h_{i-1}^2\Omega_i + \rho\delta^2\phi_1R_2^{-1}\phi_1^T + (1-\rho)\delta^2\phi_2R_2^{-1}\phi_2^T)\xi(t), \qquad (4.37)$$

where

$$\begin{aligned}
\psi &= [\psi_{ij}]_{i,j=1,..,5}, \\
\psi_{11} &= \sum_{k=1}^{3}Q_k - R_1 + S_1A_0 + A_0^TS_1^T + S_1B_2K + K^TB_2^TS_1^T \\
&\quad + \sum_{k=1}^{3}\varepsilon_kS_1D_kD_k^TS_1^T + \varepsilon_1^{-1}E_1^TE_1 + \varepsilon_3^{-1}K^TE_3^TE_3K, \\
\psi_{12} &= R_1 + A_0^TS_2^T + K^TB_2^TS_2^T + \sum_{k=1}^{3}\varepsilon_kS_1D_kD_k^TS_2^T, \\
\psi_{13} &= A_0^TS_3^T + S_1A_1 + K^TB_2^TS_3^T + \sum_{k=1}^{3}\varepsilon_kS_1D_kD_k^TS_3^T, \\
\psi_{14} &= A_0^TS_4^T + K^TB_2^TS_4^T + \sum_{k=1}^{3}\varepsilon_kS_1D_kD_k^TS_4^T, \\
\psi_{15} &= P - S_1 + A_0^TS_5^T + K^TB_2^TS_5^T + \sum_{k=1}^{3}\varepsilon_kS_1D_kD_k^TS_5^T,
\end{aligned}$$

$$
\begin{aligned}
\psi_{22} &= -(Q_2 - Q_4) - R_1 + \delta\left[M_1 + M_1^T\right] + \sum_{k=1}^{3} \varepsilon_k S_2 D_k D_k^T S_2^T, \\
\psi_{23} &= \delta\left[-M_1 + N_1^T\right] + S_2 A_1 + \sum_{k=1}^{3} \varepsilon_k S_2 D_k D_k^T S_3^T, \\
\psi_{24} &= \sum_{k=1}^{3} \varepsilon_k S_2 D_k D_k^T S_4^T, \psi_{25} = -S_2 + \sum_{k=1}^{3} \varepsilon_k S_2 D_k D_k^T S_5^T, \\
\psi_{33} &= -\sum_{k=3}^{4} Q_k + \delta\left[-N_1 - N_1^T\right] + \delta\left[M_2 + M_2^T\right] + S_3 A_1 + A_1^T S_3^T \\
&\quad + {\varepsilon_2}^{-1} E_2^T E_2 + \sum_{k=1}^{3} \varepsilon_k S_3 D_k D_k^T S_3^T, \\
\psi_{34} &= \delta\left[-M_2 + N_2^T\right] + A_1^T S_4^T + \sum_{k=1}^{3} \varepsilon_k S_3 D_k D_k^T S_4^T, \\
\psi_{35} &= -S_3 + A_1^T S_5^T + \sum_{k=1}^{3} \varepsilon_k S_3 D_k D_k^T S_5^T, \\
\psi_{44} &= -Q_1 + \delta\left[-N_2 - N_2^T\right] + \sum_{k=1}^{3} \varepsilon_k S_4 D_k D_k^T S_4^T, \\
\psi_{45} &= -S_4 + \sum_{k=1}^{3} \varepsilon_k S_4 D_k D_k^T S_5^T, \psi_{55} = \delta^2 R_2 - S_5 - S_5^T + \sum_{k=1}^{3} \varepsilon_k S_5 D_k D_k^T S_5^T, \\
\rho &= \frac{h - h_{i-1}}{\delta}, 0 \leq \rho \leq 1, \Omega_i = \begin{bmatrix} 0_{4n\times 4n} & 0_{4n\times n} \\ 0_{n\times 4n} & R_1 \end{bmatrix}.
\end{aligned}
$$

Therefore, the stability requirement for the i^{th} interval is

$$
\psi + h_{(i-1)}^2 \Omega_i + \delta^2 \phi_j R_2^{-1} \phi_j^T < 0, \quad j = 1, 2. \tag{4.38}
$$

To this end, note that, $\Omega_i \geq 0$ and the term $h_{(i-1)}^2 \Omega_i$ is maximum when $h \in [h_{(N-1)}, \bar{h}]$, the N^{th} interval. Therefore, the following condition always ensures stability of (4.28):

$$
\psi + h_{(N-1)}^2 \Omega_N + \delta^2 \phi_j R_2^{-1} \phi_j^T < 0, \quad j = 1, 2, \tag{4.39}
$$

Since the third term in (5.36) are positive definite, one may approximate them in a reduced LMI form as:

$$
\begin{bmatrix} \bar{\psi} & \delta\phi_l \\ * & -R_2 \end{bmatrix} < 0, \quad l = 1, 2 \tag{4.40}
$$

where $\bar{\psi} = \psi + h_{(i-1)}^2 \Omega_i$.
For linearization, considering S_2, S_3, S_4 and S_5 as: $S_2 = \lambda S_1$, $S_3 = \beta S_1$, $S_4 = \gamma S_1$, $S_5 = \alpha S_1$ and then, pre- and post-multiplying by $diag\left\{S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \quad S_1^{-1}\right\}$, and its transpose respectively, and subsequently adopting the change of variables

$$\begin{aligned}\bar{S}_1 &= S_1^{-1}, \bar{P} = \bar{S}_1 P \bar{S}_1^T, \bar{M}_i = \bar{S}_1 M_i \bar{S}_1^T, \bar{N}_i = \bar{S}_1 N_i \bar{S}_1^T, i = 1, 2, \bar{Q}_j = \bar{S}_1 Q_j \bar{S}_1^T, j = 1, ..4, \\ \bar{Y} &= K \bar{S}_1^T.\end{aligned}$$

One may write

$$\Pi_l + \Xi_1 + \Xi_2 + \Xi_3 < 0, \tag{4.41}$$

where

$$\begin{aligned}\Xi_1 &= \begin{bmatrix} \varepsilon_1^{-1} \bar{S}_1 E_1^T E_1 \bar{S}_1^T & 0_{5n\times 5n} \\ 0_{5n\times n} & 0_{n\times 5n} \end{bmatrix}, \Xi_2 = \begin{bmatrix} \Xi_{21} & 0_{3n\times 3n} \\ 0_{3n\times 3n} & 0_{3n\times 3n} \end{bmatrix}, \Xi_{21} = \begin{bmatrix} 0_{2n\times 2n} & 0_{2n\times n} \\ 0_{n\times 2n} & \varepsilon_2^{-1} \bar{S}_1 E_2^T E_2 \bar{S}_1^T \end{bmatrix}, \\ \Xi_3 &= \begin{bmatrix} \varepsilon_3^{-1} \bar{Y}^T E_3^T E_3 \bar{Y} & 0_{5n\times 5n} \\ 0_{5n\times n} & 0_{n\times 5n} \end{bmatrix}.\end{aligned}$$

Applying Schur complement thrice on (4.41), one obtains (4.29). □

To simplify Theorem 4.2 by eliminating free matrix variables, the following corollary is developed.

Corollary 4.2. *System (4.28) is stable if there exist $\bar{P} > 0$, $\bar{Q}_k > 0$, $\bar{R}_j > 0$, $k = 1, \ldots, 4$, $j = 1, 2$ satisfying the following LMI condition:*

$$\begin{bmatrix} \tilde{\Pi} & \check{E}_0 & \check{E}_1 & \check{E}_2 \\ * & -\varepsilon_1 I & 0 & 0 \\ * & * & -\varepsilon_2 I & 0 \\ * & * & * & -\varepsilon_3 I \end{bmatrix} < 0, \tag{4.42}$$

where

$$\begin{aligned}\tilde{\Pi} &= \bar{\Pi} + \bar{D}, \bar{\Pi} = [\bar{\Pi}_{ij}]_{i,j=1,..,5} \quad with \quad \bar{\Pi}_{11} = \check{\Pi}_{11}, \bar{\Pi}_{12} = \check{\Pi}_{12}, \bar{\Pi}_{13} = \check{\Pi}_{13}, \bar{\Pi}_{14} = \check{\Pi}_{14}, \\ \bar{\Pi}_{15} &= \check{\Pi}_{15}, \bar{\Pi}_{22} = \bar{Q}_4 - \bar{Q}_2 - \bar{R}_1 - \bar{R}_2, \bar{\Pi}_{23} = \lambda A_1 \bar{S}_1^T + \bar{R}_2, \bar{\Pi}_{24} = \check{\Pi}_{24}, \bar{\Pi}_{25} = \check{\Pi}_{25}, \\ \bar{\Pi}_{33} &= -(\bar{Q}_3 + \bar{Q}_4) - 2\bar{R}_2 + \beta A_1 \bar{S}_1^T + \beta \bar{S}_1 A_1^T, \bar{\Pi}_{34} = \gamma \bar{S}_1 A_1^T + \bar{R}_2, \bar{\Pi}_{35} = \check{\Pi}_{35}, \\ \bar{\Pi}_{44} &= -\bar{Q}_1 - \bar{R}_2, \bar{\Pi}_{45} = \check{\Pi}_{45}, \bar{\Pi}_{55} = \check{\Pi}_{55}.\end{aligned}$$

Proof. Since the last term in (4.38) is positive definite, one may reduce the stability condition in the form of a single matrix inequalities as:

$$\psi + h_{N-1}^2\Omega_N + \delta^2\phi_1 R_2^{-1}\phi_1^T + \delta^2\phi_2 R_2^{-1}\phi_2^T < 0. \tag{4.43}$$

One may write (4.43) as:

$$\bar{\psi} + \delta^2\phi_1 R_2^{-1}\phi_1^T + \delta^2\phi_2 R_2^{-1}\phi_2^T < 0, \tag{4.44}$$

where $\bar{\psi} = \psi + h_{(N-1)}^2\Omega_N$. Separating the M_1, N_1, M_2 and N_2 terms from $\bar{\psi}$, one obtains

$$\upsilon + (\delta\phi_1)I_1^T + I_1(\delta\phi_1)^T + (\delta\phi_1)R_2^{-1}(\delta\phi_1)^T + (\delta\phi_2)I_2^T + I_2(\delta\phi_2)^T + (\delta\phi_2)R_2^{-1}(\delta\phi_2)^T < 0, \tag{4.45}$$

where

$$\begin{aligned}
\upsilon &= [\upsilon_{ij}]_{i,j=1,\ldots,5}, \upsilon_{11} = \psi_{11}, \upsilon_{12} = \psi_{12}, \upsilon_{13} = \psi_{13}, \upsilon_{14} = \psi_{14}, \upsilon_{15} = \psi_{15},\\
\upsilon_{22} &= -(Q_2 - Q_4) - R_1 + \sum_{k=1}^{3}\varepsilon_k S_2 D_k D_k^T S_2^T, \upsilon_{23} = S_2 A_1 + \sum_{k=1}^{3}\varepsilon_k S_2 D_k D_k^T S_3^T, \upsilon_{24} = \psi_{24},\\
\upsilon_{25} &= \psi_{25}, \upsilon_{33} = -\sum_{k=3}^{4} Q_k + S_3 A_1 + A_1^T S_3^T + \varepsilon_2{}^{-1} E_2^T E_2 + \sum_{k=1}^{3}\varepsilon_k S_3 D_k D_k^T S_3^T,\\
\upsilon_{34} &= A_1^T S_4^T + \sum_{k=1}^{3}\varepsilon_k S_3 D_k D_k^T S_4^T, \upsilon_{35} = \psi_{35}, \upsilon_{44} = -Q_1 + \sum_{k=1}^{3}\varepsilon_k S_4 D_k D_k^T S_4^T, \upsilon_{45} = \psi_{45},\\
\upsilon_{55} &= \psi_{55}, I_1 = \begin{bmatrix} 0 & I & -I & 0 & 0\end{bmatrix}^T, I_2 = \begin{bmatrix} 0 & 0 & I & -I & 0\end{bmatrix}^T.
\end{aligned}$$

One can write (4.45) as:

$$\upsilon + (\delta\phi_1 + I_1R_2)\,R_2^{-1}(\delta\phi_1 + I_1R_2)^T - I_1R_2I_1^T + (\delta\phi_2 + I_2R_2)\,R_2^{-1}(\delta\phi_2 + I_2R_2)^T - I_2R_2I_2^T < 0. \tag{4.46}$$

Further, following Lemma 1.2, substituting the free variables as $M_i = M_i{}^T = -N_i = -N_i{}^T = -\delta^{-1}R_2$, the above stability condition yields.

$$\bar{\upsilon} < 0, \tag{4.47}$$

where

$$\bar{\upsilon} = [\bar{\upsilon}_{ij}]_{i,j=1,\ldots,5}, \bar{\upsilon}_{11} = \upsilon_{11}, \bar{\upsilon}_{12} = \upsilon_{12}, \bar{\upsilon}_{13} = \upsilon_{13}, \bar{\upsilon}_{14} = \upsilon_{14}, \bar{\upsilon}_{15} = \upsilon_{15},$$

$$\begin{aligned}
\bar{\upsilon}_{22} &= -(Q_2 - Q_4) - R_1 - R_2 + \sum_{k=1}^{3} \varepsilon_k S_2 D_k D_k^T S_2^T, \bar{\upsilon}_{23} = S_2 A_1 + R_2 + \sum_{k=1}^{3} \varepsilon_k S_2 D_k D_k^T S_3^T, \\
\bar{\upsilon}_{24} &= \upsilon_{24}, \bar{\upsilon}_{25} = \upsilon_{25}, \bar{\upsilon}_{33} = -\sum_{k=3}^{4} Q_k + S_3 A_1 + A_1^T S_3^T + \varepsilon_2^{-1} E_2^T E_2 - 2R_2 + \sum_{k=1}^{3} \varepsilon_k S_3 D_k D_k^T S_3^T, \\
\bar{\upsilon}_{34} &= A_1^T S_4^T + R_2 + \sum_{k=1}^{3} \varepsilon_k S_3 D_k D_k^T S_4^T, \bar{\upsilon}_{35} = \upsilon_{35}, \bar{\upsilon}_{44} = -Q_1 - R_2 + \sum_{k=1}^{3} \varepsilon_k S_4 D_k D_k^T S_4^T, \\
\bar{\upsilon}_{45} &= \upsilon_{45}, \bar{\upsilon}_{55} = \upsilon_{55}.
\end{aligned}$$

For linearization, considering S_2, S_3, S_4 and S_5 as: $S_2 = \lambda S_1$, $S_3 = \beta S_1$, $S_4 = \gamma S_1$, $S_5 = \alpha S_1$ and then, pre- and post-multiplying by $diag\left\{S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \quad S_1^{-1}\right\}$, and its transpose respectively, and subsequently adopting the change of variables

$$\begin{aligned}
\bar{S}_1 &= S_1^{-1}, \bar{P} = \bar{S}_1 P \bar{S}_1^T, \bar{M}_i = \bar{S}_1 M_i \bar{S}_1^T, \bar{N}_i = \bar{S}_1 N_i \bar{S}_1^T, i = 1, 2, \bar{Q}_j = \bar{S}_1 Q_j \bar{S}_1^T, j = 1, ..4, \\
\bar{Y} &= K \bar{S}_1^T.
\end{aligned}$$

one obtains (5.39). □

In some cases, the uncertain system may be described in simple fashion as:

$$\dot{x}(t) = A_0(t)x(t) + A_1(t)x(t-h) + B_2 u(t), \tag{4.48}$$

where $A_0(t)$ and $A_1(t)$ are the matrices with time-varying uncertainty, and B_2 is a constant matrix with appropriate dimension

$$A_0(t) = A_0 + \Delta A_0(t), A_1(t) = A_1 + \Delta A_1(t), \tag{4.49}$$

where $\Delta A_0(t)$, $\Delta A_1(t)$ are uncertain components of the nominal matrices A_0, A_1 respectively. The uncertain matrices are norm bounded and can be decomposed in simple fashion as:

$$\begin{bmatrix} \Delta A_0(t) & \Delta A_1(t) \end{bmatrix} = DF(t)E, \tag{4.50}$$

where D and E are appropriate dimensional constant matrices, and $F(t)$ satisfies $F^T(t)F(t) \leq I$. Using (4.2) in (4.48), one obtains the closed loop system as:

$$\dot{x}(t) = A_0(t)x(t) + A_1(t)x(t-h) + B_2 K x(t), \tag{4.51}$$

To design K for (4.51), the following stabilization criterion is developed.

Corollary 4.3. *System (4.51) is stable if there exist matrices $\bar{P} > 0$, $\bar{Q}_j > 0$, $j = 1, \ldots, 4$, $\bar{R}_i > 0$ and arbitrary matrices $\bar{S}_l$, $\bar{M}_i$, $\bar{N}_i$, $l = 1 \ldots 5$, $i = 1, 2$, that satisfy the following LMI:*

$$\begin{bmatrix} \nabla_k & \tilde{E} \\ * & -\varepsilon I \end{bmatrix} < 0, \tag{4.52}$$

where

$$\begin{aligned} \nabla_k &= \begin{bmatrix} \bar{\nabla} & \delta\bar{\phi}_k \\ * & -\bar{R}_2 \end{bmatrix}, k = 1, 2, \bar{\nabla} = \check{\Pi} + \tilde{D}, \tilde{D} = \varepsilon \hat{D}^T \hat{D}, \hat{D} = \begin{bmatrix} D^T & \lambda D^T & \beta D^T & \gamma D^T & \alpha D^T \end{bmatrix}, \\ \tilde{E} &= \begin{bmatrix} E\bar{S}_1^T & 0 & E\bar{S}_1^T & 0 & 0 \end{bmatrix}^T, \check{\Pi} \text{ is mentioned in Theorem 4.2.} \end{aligned}$$

Proof. To prove this corollary, consider the LK functional same as (4.30) and obtain the derivative of the functional as in the case of Theorem 4.2. Instead of replacing $\dot{x}(t)$ by directly using (4.51), consider the quadratic formulation of the system dynamics (4.51) as:

$$\begin{aligned} &2\left\{x^T(t)S_1 + x^T(t - h_{i-1})S_2 + x^T(t - h)S_3 + x^T(t - h_i)S_4 + \dot{x}^T(t)S_5\right\} \\ &\quad \times \left\{-\dot{x}(t) + A_0(t)x(t) + A_1(t)x(t - h) + B_2Kx(t)\right\} = 0, \end{aligned} \tag{4.53}$$

where $S_k, k = 1, \ldots, 5$ are arbitrary matrices of appropriate dimensions. Following (4.49), (4.50) and Lemma 2.2, one may write

$$\begin{aligned} &2\xi^T(t)\begin{bmatrix} S_1^T & S_2^T & S_3^T & S_4^T & S_5^T \end{bmatrix}^T DF(t)\begin{bmatrix} Ex(t) & Ex(t - h) \end{bmatrix} \\ &\quad \leq \varepsilon\xi^T(t)\check{D}^T\check{D}\xi(t) + \varepsilon^{-1}\xi^T(t)\grave{E}^T\grave{E}\xi(t), \end{aligned} \tag{4.54}$$

where

$$\begin{aligned} \xi(t) &= \begin{bmatrix} x^T(t) & x^T(t - h_{(i-1)}) & x^T(t - h) & x^T(t - h_i) & \dot{x}^T(t) \end{bmatrix}^T, \\ \check{D} &= \begin{bmatrix} D^T S_1^T & D^T S_2^T & D^T S_3^T & D^T S_4^T & D^T S_5^T \end{bmatrix}^T, \grave{E} = \begin{bmatrix} E & 0 & E & 0 & 0 \end{bmatrix}^T, \end{aligned}$$

Adding (4.53) to (4.31) by replacing the uncertain term by R.H.S. of (4.54), one obtains

$$\begin{aligned}\dot{V}_i(t) \leq\ & 2\xi^T(t)\begin{bmatrix} S_1^T & S_2^T & S_3^T & S_4^T & S_5^T \end{bmatrix}^T \{-\dot{x}(t) + (A_0 + B_2K)x(t) + A_1x(t-h)\} \\ & + \varepsilon\xi^T(t)\check{D}^T\check{D}\xi(t) + \varepsilon^{-1}\xi^T(t)\grave{E}^T\grave{E}\xi(t) + 2x^T(t)P\dot{x}(t) + \sum_{k=1}^{3} x^T(t)Q_kx(t) \\ & - x^T(t-h_{(i-1)})(Q_2 - Q_4)x(t-h_{(i-1)}) - \sum_{k=3}^{4} x^T(t-h)Q_kx(t-h) \\ & - x^T(t-h_i)Q_1x(t-h_i) + \dot{x}^T(t)\left\{h_{(i-1)}^2R_1 + \delta^2R_2\right\}\dot{x}(t) \\ & - h_{(i-1)}\int_{t-h_{(i-1)}}^{t} \dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta - \delta\int_{t-h_i}^{t-h_{(i-1)}} \dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta. \end{aligned} \tag{4.55}$$

Replacing the integral terms of (4.55) by following Lemma 1.2 as in case of Theorem 4.2, one obtains

$$\dot{V}_i(t) \leq \xi^T(t)(\Xi + h_{i-1}^2\Omega_i + \rho\delta^2\phi_1R_2^{-1}\phi_1^T + (1-\rho)\delta^2\phi_2R_2^{-1}\phi_2^T)\xi(t), \tag{4.56}$$

where

$$\begin{aligned}
\Xi &= [\Xi_{ij}]_{i,j=1,..,5}, \\
\Xi_{11} &= \sum_{k=1}^{3} Q_k - R_1 + S_1A_0 + A_0^TS_1^T + S_1B_2K + K^TB_2^TS_1^T + \varepsilon S_1DD^TS_1^T + \varepsilon^{-1}E^TE, \\
\Xi_{12} &= R_1 + A_0^TS_2^T + K^TB_2^TS_2^T + \varepsilon S_1DD^TS_2^T, \\
\Xi_{13} &= A_0^TS_3^T + S_1A_1 + K^TB_2^TS_3^T + \varepsilon^{-1}E^TE + \varepsilon S_1DD^TS_3^T, \\
\Xi_{14} &= A_0^TS_4^T + K^TB_2^TS_4^T + \varepsilon S_1DD^TS_4^T, \Xi_{15} = P - S_1 + A_0^TS_5^T + K^TB_2^TS_5^T + \varepsilon S_1DD^TS_5^T, \\
\Xi_{22} &= -(Q_2 - Q_4) - R_1 + \delta\left[M_1 + M_1^T\right] + \varepsilon S_2DD^TS_2^T, \\
\Xi_{23} &= \delta\left[-M_1 + N_1^T\right] + S_2A_1 + \varepsilon S_2DD^TS_3^T, \Xi_{24} = \varepsilon S_2DD^TS_4^T, \Xi_{25} = -S_2 + \varepsilon S_2DD^TS_5^T, \\
\Xi_{33} &= -\sum_{k=3}^{4} Q_k + \delta\left[-N_1 - N_1^T + M_2 + M_2^T\right] + +S_3A_1 + A_1^TS_3^T + \varepsilon^{-1}E^TE + \varepsilon S_3DD^TS_3^T, \\
\Xi_{34} &= \delta\left[-M_2 + N_2^T\right] + A_1^TS_4^T + \varepsilon S_3DD^TS_4^T, \Xi_{35} = -S_3 + A_1^TS_5^T + \varepsilon S_3DD^TS_5^T, \\
\Xi_{44} &= -Q_1 + \delta\left[-N_2 - N_2^T\right] + \varepsilon S_4DD^TS_4^T, \Xi_{45} = -S_4 + \varepsilon S_4DD^TS_5^T, \\
\Xi_{55} &= \delta^2R_2 - S_5 - S_5^T + \varepsilon S_5DD^TS_5^T, \rho = \frac{h - h_{i-1}}{\delta}, 0 \leq \rho \leq 1, \Omega_i = \begin{bmatrix} 0_{4n\times 4n} & 0_{4n\times n} \\ 0_{n\times 4n} & R_1 \end{bmatrix}.
\end{aligned}$$

The stability requirement for the i^{th} interval becomes

$$\Xi + h_{(i-1)}^2 \Omega_i + \delta^2 \phi_j R_2^{-1} \phi_j^T < 0, \quad j = 1, 2. \tag{4.57}$$

The following condition ensures stability of (4.51):

$$\Xi + h_{(N-1)}^2 \Omega_N + \delta^2 \phi_j R_2^{-1} \phi_j^T < 0, \quad j = 1, 2. \tag{4.58}$$

One may approximate (4.58) in a reduced LMI form as:

$$\begin{bmatrix} \bar{\Xi} & \delta\phi_l \\ * & -R_2 \end{bmatrix} < 0, \quad l = 1, 2, \tag{4.59}$$

where $\bar{\Xi} = \Xi + h_{(i-1)}^2 \Omega_i$.
For linearization, considering S_2, S_3, S_4 and S_5 as: $S_2 = \lambda S_1$, $S_3 = \beta S_1$, $S_4 = \gamma S_1$, $S_5 = \alpha S_1$, and then, pre- and post-multiplying (4.59) by $diag \left\{ S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \right\}$ and its transpose respectively, and subsequently adopting the change of variables

$$\begin{aligned} \bar{S}_1 &= S_1^{-1}, \bar{P} = \bar{S}_1 P \bar{S}_1^T, \bar{M}_i = \bar{S}_1 M_i \bar{S}_1^T, \bar{N}_i = \bar{S}_1 N_i \bar{S}_1^T, i = 1, 2, \bar{Q}_j = \bar{S}_1 Q_j \bar{S}_1^T, j = 1, ..4, \\ \bar{Y} &= K \bar{S}_1^T. \end{aligned}$$

One may write

$$\xi^T(t)(\nabla_k + \varepsilon^{-1} \tilde{E}^T \tilde{E})\xi(t) < 0, \tag{4.60}$$

where ∇_k and $\tilde{E}$ are mentioned in (4.52). Applying Schur complement on (4.60), one obtains (4.52). □

By eliminating the free matrix variables from Corollary 4.3, the following result is developed.

Corollary 4.4. *System (4.51) is stable if there exist matrices $\bar{P} > 0$, $\bar{Q}_j > 0$, $j = 1, \ldots, 4$, $\bar{R}_i > 0$, $i = 1, 2,$and arbitrary matrices $\bar{S}_l$, $l = 1 \ldots 5$ that satisfy the following LMI:*

$$\begin{bmatrix} \tilde{\nabla} & \tilde{E} \\ * & -\varepsilon I \end{bmatrix} < 0, \tag{4.61}$$

where

$$\begin{aligned} \tilde{\nabla} &= \bar{\Pi} + \tilde{D}, \tilde{D} = \varepsilon \hat{D}^T \hat{D}, \hat{D} = \begin{bmatrix} D^T & \lambda D^T & \beta D^T & \gamma D^T & \alpha D^T \end{bmatrix}, \\ \tilde{E} &= \begin{bmatrix} E\bar{S}_1^T & 0 & E\bar{S}_1^T & 0 & 0 \end{bmatrix}^T, \bar{\Pi} \text{ is mentioned in Corollary 4.2.} \end{aligned}$$

Proof. Since the last term in (4.58) is positive definite, one may reduce the stability condition in the form of a single matrix inequalities as:

$$\Xi + h_{N-1}^2\Omega_N + \delta^2\phi_1 R_2^{-1}\phi_1^T + \delta^2\phi_2 R_2^{-1}\phi_2^T < 0. \tag{4.62}$$

one may write (4.62) as:

$$\bar{\Xi} + \delta^2\phi_1 R_2^{-1}\phi_1^T + \delta^2\phi_2 R_2^{-1}\phi_2^T < 0, \tag{4.63}$$

where $\bar{\Xi} = \Xi + h_{(N-1)}^2\Omega_N$. Separating the M_1, N_1, M_2 and N_2 terms from $\bar{\Xi}$, one obtains

$$\Sigma+(\delta\phi_1)I_1^T+I_1(\delta\phi_1)^T+(\delta\phi_1)R_2^{-1}(\delta\phi_1)^T+(\delta\phi_2)I_2^T+I_2(\delta\phi_2)^T+(\delta\phi_2)R_2^{-1}(\delta\phi_2)^T < 0, \tag{4.64}$$

where

$$\begin{aligned}
\Sigma &= [\Sigma_{ij}]_{i,j=1,\ldots,5}, \Sigma_{11} = \Xi_{11}, \Sigma_{12} = \Xi_{12}, \Sigma_{13} = \Xi_{13}, \Sigma_{14} = \Xi_{14}, \Sigma_{15} = \Xi_{15},\\
\Sigma_{22} &= -(Q_2 - Q_4) - R_1 + \varepsilon S_2 DD^T S_2^T, \Sigma_{23} = S_2 A_1 + \varepsilon S_2 DD^T S_3^T, \Sigma_{24} = \Xi_{24},\\
\Sigma_{25} &= \Xi_{25}, \Sigma_{33} = -\sum_{k=3}^{4} Q_k + S_3 A_1 + A_1^T S_3^T + \varepsilon^{-1}E^T E + \varepsilon S_3 DD^T S_3^T,\\
\Sigma_{34} &= A_1^T S_4^T + \varepsilon S_3 DD^T S_4^T, \Sigma_{35} = \Xi_{35}, \Sigma_{44} = -Q_1 + \varepsilon S_4 DD^T S_4^T, \Sigma_{45} = \Xi_{45},\\
\Sigma_{55} &= \Xi_{55}, I_1 = \begin{bmatrix} 0 & I & -I & 0 & 0 \end{bmatrix}^T, I_2 = \begin{bmatrix} 0 & 0 & I & -I & 0 \end{bmatrix}^T.
\end{aligned}$$

One can write (4.64) as:

$$\Sigma+(\delta\phi_1 + I_1R_2)\, R_2^{-1}(\delta\phi_1 + I_1R_2)^T - I_1R_2I_1^T + (\delta\phi_2 + I_2R_2)\, R_2^{-1}(\delta\phi_2 + I_2R_2)^T - I_2R_2I_2^T < 0. \tag{4.65}$$

Further, following Lemma 1.2, substituting the free variables as $M_i = {M_i}^T = -N_i = -{N_i}^T = -\delta^{-1}R_2$, the above stability condition yields.

$$\bar{\Sigma} < 0, \tag{4.66}$$

where

$$\begin{aligned}
\bar{\Sigma} &= [\bar{\Sigma}_{ij}]_{i,j=1,\ldots,5}, \bar{\Sigma}_{11} = \Sigma_{11}, \bar{\Sigma}_{12} = \Sigma_{12}, \bar{\Sigma}_{13} = \Sigma_{13}, \bar{\Sigma}_{14} = \Sigma_{14}, \bar{\Sigma}_{15} = \Sigma_{15},\\
\bar{\Sigma}_{22} &= -(Q_2 - Q_4) - R_1 - R_2 + \varepsilon S_2 DD^T S_2^T, \bar{\Sigma}_{23} = S_2 A_1 + R_2 + \varepsilon S_2 DD^T S_3^T,\\
\bar{\Sigma}_{24} &= \Sigma_{24}, \bar{\Sigma}_{25} = \Sigma_{25}, \bar{\Sigma}_{33} = -\sum_{k=3}^{4} Q_k + S_3 A_1 + A_1^T S_3^T + \varepsilon^{-1}E^T E - 2R_2 + \varepsilon S_3 DD^T S_3^T,
\end{aligned}$$

$$
\begin{aligned}
\bar{\Sigma}_{34} &= A_1^T S_4^T + R_2 + \varepsilon S_3 D D^T S_4^T, \bar{\Sigma}_{35} = \Sigma_{35}, \bar{\Sigma}_{44} = -Q_1 - R_2 + \varepsilon S_4 D D^T S_4^T, \\
\bar{\Sigma}_{45} &= \Sigma_{45}, \bar{\Sigma}_{55} = \Sigma_{55}.
\end{aligned}
$$

For linearization, considering S_2, S_3, S_4 and S_5 as: $S_2 = \lambda S_1$, $S_3 = \beta S_1$, $S_4 = \gamma S_1$, $S_5 = \alpha S_1$, and then, pre- and post-multiplying (4.66) by $diag\left\{ S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \right\}$ and its transpose respectively, and subsequently adopting the change of variables

$$
\begin{aligned}
\bar{S}_1 &= S_1^{-1}, \bar{P} = \bar{S}_1 P \bar{S}_1^T, \bar{M}_i = \bar{S}_1 M_i \bar{S}_1^T, \bar{N}_i = \bar{S}_1 N_i \bar{S}_1^T, i = 1, 2, \bar{Q}_j = \bar{S}_1 Q_j \bar{S}_1^T, j = 1, ..4, \\
\bar{Y} &= K \bar{S}_1^T.
\end{aligned}
$$

one obtains (4.61). $\square$

The next section presents some numerical examples to validate the developed criteria in this section.

4.3.3 Numerical examples

Example 4.3. Consider an uncertain system of the form (4.51) with

$$
A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, D_1 = D_2 = 0.2I, D_3 = 0, E_1 = E_2 = I, E_3 = 0
$$

Using the proposed stabilization criterion based on decomposition technique, one is expected to get maximum delay bound value at $N = 2$. At $N = 2$, comparison of maximum tolerable delay bound for this system is made in the Table 4.3 which is shown below. Note that, for this example, Corollary 4.4 yields better result than Theorem 4.2 since several of the uncertain terms in this approach are treated through a single normalized uncertainty matrix, $F(t)$ only. The conservatism is less since, one has to bound only a single term rather than three terms following the approach in Theorem 4.2.

Example 4.4. Consider another example in the form (4.51) with

$$
A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} -2 & -0.5 \\ 0 & -1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, D_1 = D_2 = 0.2I, D_3 = 0, E_1 = E_2 = I, E_3 = 0
$$

For this case also, the controller is designed by setting $N = 2$. A comparison of maximum tolerable delay bound for this system is done in the Table 4.4. From the comparison, it is

Table 4.3: Comparison of delay bound ($\bar{h}$)

Methods	$\bar{h}$	Controller gain (K)
[90]	0.2250	-
[19]	0.3346	-
[126]	1.3	[-2.1485 -5.6948]
Theorem 4.2	1.2498	[-2246.5 -5142.5]
Corollary 4.4	1.3870	[-370.3688 -867.3798]

clear that the proposed decomposition approach customarily gives less conservative result. Similar to the previous example, Corollary 4.4 yields better result here as well.

Table 4.4: Comparison of delay bound ($\bar{h}$)

Methods	$\bar{h}$	Controller gain (K)
[90]	0.2716	-
[118]	0.3015	-
[111]	0.4500	[-4.8122 -7.7129]
[34]	0.5865	[-0.3155 -4.4417]
[126]	0.6900	[-23.2572 -26.1488]
Theorem 4.2	0.6905	[6541.9 -5451.4]
Corollary 4.4	0.7605	[-4228.8 -3048.3]

4.4 Chapter summary

This section summarizes the contributions made in this chapter.

- A Less conservative stabilization criterion for systems with single delay has been proposed.
- The decomposition technique proposed in Chapter 2 is utilized to derive such less conservative stabilization criteria using static state feedback controller for systems with constant delay.
- As the proposed decomposition technique does not require a complete LK functional to derive the stabilization criterion, it results in simple LMI condition.

- A sufficient stabilization criterion is derived by decomposing the whole delay range into several intervals and drawing a single one out of them by defining a simple multiple LK functional.
- Though the decomposition technique gives a sufficient stabilization criteria, they are less conservative than the existing approach.
- The resulting criterion is independent of the number of decomposition of the delay interval as a result of which a finite-dimensional LMI is formulated.
- The complexity of the stabilization criterion does not increase with increase in number of decomposition.
- The stabilization criterion of the nominal system has been easily extended for uncertain systems.
- Numerical examples are presented to show the effectiveness of the proposed criteria, which are less conservative than the existing ones while being computationally efficient for using lesser number of LMI variables.

CHAPTER 5

Stabilization of systems with input-delay

In this chapter, improved delay-dependent robust stabilization criteria for systems with input-delay using state feedback controllers are proposed. To derive such criteria, suitable Lyapunov-Krasovskii functional is chosen and free-weighted matrix variable approach is employed. First, a simple static state feedback robust stabilization criterion is formulated in terms of LMI. The major issue in such type of stabilization problem is the linearization of nonlinear terms in the criterion. To address such an issue, a simple linearization technique is used. Next, the decomposition approach proposed in Chapter 2 is implemented to design less conservative robust static state feedback controller. Stabilizability of such systems with a PI-type state feedback controller is then investigated. The PI-type controller includes both proportional and integral control actions whereas static state feedback controller has only proportional control action. Due to the involvement of the integral control action, the dimension of the overall system increases along with the controller dynamics as well. This might introduce flexibility to control designers to search the control gain parameters in a higher dimensional space with more number of parameters. This way, robustness improvement using the PI-type controller is addressed in this chapter. Some numerical examples are considered to demonstrate the effectiveness of the proposed stabilization criteria for input delay system.

5.1 Introduction

Time delays in control inputs are often encountered in feedback control systems. Given the development in the previous chapter, one may wonder what are the differences in stabilization of systems with state delay and input delay. The differences involved are as follows: (i) In systems with state delay, the delay is involved with known parameters of the system whereas in input delay, the delay is involved with unknown parameter, i.e. the controller gain K that is to be designed, (ii) For systems with input delay, specifically with constant known delay, a model transformation exists [4, 76, 85] that transforms the system into a non-time-delay one. Exploiting this transformation a complex structured controller has been designed in [19,67,110,144]. The same has further been investigated in [14,117,174,179]. These issues in stabilization of input-delay systems attracts researchers for further investigation.

Using reduction method [85, 110], an input-delayed system is reduced to a delay-free ordinary system and a controller is designed for the reduced system. Though this approach overcomes the problems of the conventional Smith predictor method but the approach is conservative and it requires the exact value of the time-delay. An LMI based approach using controller with memory proposed in [173] that does not require the exact value of the delay. The same author further introduces a new state transformation approach in [174] to reduce the conservatism in previous design. Under this approach, the controller design only requires to know the change in interval of the input delays rather than the exact values. However, this method is conservative and uses memory type controller which is complicated to realize while implementation. By using a simple state feedback controller, an integral inequality approach is proposed in [179] to obtain a less conservative criterion. For further reduction in conservatism, a state transformation approach of [174] with descriptor system approach are used in [14] to obtain the stabilization criterion in terms of LMIs. However, once again, the drawback of this approach is that it uses memory based controller which is difficult to realize. This motivates to investigate on stabilization of a input-delay system using simple LK functional.

In this chapter, state feedback stabilization criteria for uncertain systems with input-delay have been developed. From the robust stabilization criteria, one can easily obtain stabilization criteria for nominal systems with input-delay. To obtain the desired criteria, a free weighted matrix variable method is used to get the delay-dependent criterion via LMI formulation. During the process of obtaining the criteria, non-linear terms are evolved due to unknown controller parameter K. To handle this problem, a simple linearization technique is used. And a tighter delay bound is used for integral approximation in the derivative of the functional. First, using the above techniques static state feedback stabilization criterion is derived. Next, to reduce conservatism, the proposed decomposition technique in Chapter 2 is used to design static state feedback controller. Finally, a PI (Proportional and Integral)-type controller is used to improve the tolerable input-delay bound.

5.2 System description

Consider an uncertain system with input delay

$$\dot{x}(t) = A_0(t)x(t) + B_1(t)u(t) + B_2(t)u(t-h), \tag{5.1}$$

where $x(t) \in \Re^n$ is the state, and $u(t) \in \Re^n$ is the control input; h is constant delay satisfying $0 \leq h_1 \leq h \leq h_2$, $\bar{h} = h_2 - h_1$; $A_0(t)$, $B_1(t)$ and $B_2(t)$ are matrices with time-varying uncertainties and can be decomposed as:

$$A_0(t) = A_0 + \Delta A_0(t), B_1(t) = B_1 + \Delta B_1(t), B_2(t) = B_2 + \Delta B_2(t), \tag{5.2}$$

where A_0, B_1 and B_2 are constant matrices of appropriate dimensions, and $\Delta A_0(t)$, $\Delta B_1(t)$ and $\Delta B_2(t)$ are unknown perturbed matrices representing time-varying parameter uncertainties in the system model. Assuming that the uncertainties are norm bounded and can be represented as:

$$\begin{bmatrix} \Delta A_0(t) & \Delta B_1(t) & \Delta B_2(t) \end{bmatrix} = \begin{bmatrix} D_0F(t)E_0 & D_1F(t)E_1 & D_2F(t)E_2 \end{bmatrix}, \tag{5.3}$$

where D_0, D_1, D_2, E_0, E_1 and E_2 are appropriately dimensioned constant matrices, and $F(t)$ is an unknown real and possibly time-varying matrix with is Lebesgue measurable elements satisfying $F^T(t)F(t) \leq I$, $\forall t$.

5.3 Simple stabilization using memory less controller

This section presents static state feedback controller based robust stabilization using simple LK functional. The objective is to derive a stabilization criterion for (5.1) using a controller of the form

$$u(t) = Kx(t), \tag{5.4}$$

where K is the control gain matrix of appropriate dimension. The closed loop system becomes

$$\dot{x}(t) = A_0(t)x(t) + B_1(t)Kx(t) + B_2(t)Kx(t-h), \tag{5.5}$$

The following lemma will be used to derive main stabilization criterion in this section.

Lemma 5.1. *For any arbitrary matrix S_1, S_2, S_3, S_4 and S_5, the following condition holds:*

$$\begin{aligned} 2X^T(t)\begin{bmatrix} S_1^T & S_2^T & S_3^T & S_4^T & S_5^T \end{bmatrix}^T \times \{-\dot{x}(t) + A_0x(t) + B_1Kx(t) + B_2Kx(t-h)\} \\ + \sum_{k=0}^{2} \varepsilon_k X^T(t)\hat{D}_k^T\hat{D}_kX(t) + \varepsilon_0^{-1}x^T(t)E_0^TE_0x(t) + \varepsilon_1^{-1}x^T(t)K^TE_1^TE_1Kx(t) \\ + \varepsilon_2^{-1}x^T(t-h)K^TE_2^TE_2Kx(t-h) \geq 0, \end{aligned} \tag{5.6}$$

where

$$\begin{aligned} X(t) &= \begin{bmatrix} x^T(t) & x^T(t-h_1) & x^T(t-h) & x^T(t-h_2) & \dot{x}^T(t) \end{bmatrix}^T, \\ \hat{D}_k &= \begin{bmatrix} D_k^TS_1^T & D_k^TS_2^T & D_k^TS_3^T & D_k^TS_4^T & D_k^TS_5^T \end{bmatrix}. \end{aligned}$$

Proof. Following (5.5), one can write

$$2X^T(t)\begin{bmatrix} S_1^T & S_2^T & S_3^T & S_4^T & S_5^T \end{bmatrix}^T\{-\dot{x}(t) + A_0(t)x(t) + B_1(t)Kx(t) + B_2(t)Kx(t-h)\} = 0. \tag{5.7}$$

Substituting (5.2) and (5.3) in (5.7) and by following Lemma 2.2, the uncertain terms can be written as:

$$\begin{aligned} &2X^T(t)\begin{bmatrix} S_1^T & S_2^T & S_3^T & S_4^T & S_5^T \end{bmatrix}^T D_0F(t)E_0x(t) \\ &\leq \varepsilon_0X^T(t)\hat{D}_0^T\hat{D}_0X(t) + \varepsilon_0^{-1}x^T(t)E_0^TE_0x(t). \end{aligned} \tag{5.8}$$

$$\begin{aligned} &2X^T(t)\begin{bmatrix} S_1^T & S_2^T & S_3^T & S_4^T & S_5^T \end{bmatrix}^T D_1F(t)E_1Kx(t) \\ &\leq \varepsilon_1X^T(t)\hat{D}_1^T\hat{D}_1X(t) + \varepsilon_1^{-1}x^T(t)K^TE_1^TE_1Kx(t). \end{aligned} \tag{5.9}$$

$$\begin{aligned} &2X^T(t)\begin{bmatrix} S_1^T & S_2^T & S_3^T & S_4^T & S_5^T \end{bmatrix}^T D_2F(t)E_2Kx(t-h) \\ &\leq \varepsilon_2X^T(t)\hat{D}_2^T\hat{D}_2X(t) + \varepsilon_2^{-1}x^T(t-h)K^TE_2^TE_2Kx(t-h). \end{aligned} \tag{5.10}$$

Replacing the terms in (5.8), (5.9) and (5.10) by the uncertain terms of (5.7), one obtains (5.6). □

The above lemma is used to derive the following robust stabilization criterion for (5.5).

5.3.1 Stabilization criterion

A controller of the form (5.4) is designed for (5.1) using the stabilization criterion presented below.

Theorem 5.1. *System (5.1) is stable if, for arbitrarily chosen real scalar quantities such as: λ, β, γ and α, there exist matrices $\bar{P} > 0$, $\bar{Q}_j > 0$, $j = 1, \ldots, 4$, $\bar{R}_i > 0$, scalars ε_k, $k = 0, \ldots, 2$. and arbitrary matrices $\bar{S}_1$, $\bar{M}_i$, $\bar{N}_i$, $i = 1, 2$, that satisfy the following LMI:*

$$\begin{bmatrix} \Theta_l & \tilde{E}_0 & \tilde{E}_1 & \tilde{E}_2 \\ * & -\varepsilon_0 I & 0 & 0 \\ * & * & -\varepsilon_1 I & 0 \\ * & * & * & -\varepsilon_2 I \end{bmatrix} < 0, \quad l = 1, 2, \tag{5.11}$$

where

$$\begin{aligned}
\Theta_l &= \begin{bmatrix} \bar{\Theta} & \bar{\Phi}_l \\ * & -\bar{R}_2 \end{bmatrix}, \quad l = 1, 2, \quad \bar{\Theta} = \Theta + \bar{D}, \quad \bar{D} = \sum_{k=0}^{2} \varepsilon_k \bar{D}_k^T \bar{D}_k, \\
\bar{D}_k &= \begin{bmatrix} D_k^T & \lambda D_k^T & \beta D_k^T & \gamma D_k^T & \alpha D_k^T \end{bmatrix}, \bar{\Phi}_1 = \begin{bmatrix} 0 & \bar{M}_1^T & \bar{N}_1^T & 0 & 0 \end{bmatrix}^T, \\
\bar{\Phi}_2 &= \begin{bmatrix} 0 & 0 & \bar{M}_2^T & \bar{N}_2^T & 0 \end{bmatrix}^T, \Theta = [\Theta_{ij}]_{i,j=1,..,5} \quad \textit{with} \\
\Theta_{11} &= \bar{Q}_1 + \bar{Q}_2 + \bar{Q}_3 - \bar{R}_1 + A_0 \bar{S}_1^T + \bar{S}_1 A_0^T + B_1 \bar{Y} + \bar{Y}^T B_1{}^T, \\
\Theta_{12} &= \bar{R}_1 + \lambda \bar{S}_1 A_0^T + \lambda \bar{Y}^T B_1^T, \Theta_{13} = \beta \bar{S}_1 A_0^T + \beta \bar{Y}^T B_1^T + B_2 \bar{Y}, \Theta_{14} = \gamma \bar{S}_1 A_0^T + \gamma \bar{Y}^T B_1^T, \\
\Theta_{15} &= \bar{P} - \bar{S}_1^T + \alpha \bar{S}_1 A_0^T + \alpha \bar{Y}^T B_1^T, \Theta_{22} = \bar{Q}_4 - \bar{Q}_1 - \bar{R}_1 + \bar{h}^{-1}(\bar{M}_1 + \bar{M}_1^T), \\
\Theta_{23} &= \lambda B_2 \bar{Y} + \bar{h}^{-1}(-\bar{M}_1 + \bar{N}_1^T), \Theta_{24} = 0, \Theta_{25} = -\lambda \bar{S}_1^T, \\
\Theta_{33} &= -(\bar{Q}_3 + \bar{Q}_4) + \beta B_2 \bar{Y} + \beta \bar{Y}^T B_2^T + \bar{h}^{-1}(-\bar{N}_1 - \bar{N}_1^T) + \bar{h}^{-1}(\bar{M}_2 + \bar{M}_2^T), \\
\Theta_{34} &= \gamma \bar{Y}^T B_2^T + \bar{h}^{-1}(-\bar{M}_2 + \bar{N}_2^T), \Theta_{35} = -\beta \bar{S}_1^T + \alpha \bar{Y}^T B_2^T, \Theta_{44} = -\bar{Q}_2 + \bar{h}^{-1}(-\bar{N}_2 - \bar{N}_2^T), \\
\Theta_{45} &= -\gamma \bar{S}_1^T, \Theta_{55} = -\alpha \bar{S}_1 - \alpha \bar{S}_1^T + \left\{ h_1^2 \bar{R}_1 + \bar{R}_2 \right\}, \tilde{E}_0 = \begin{bmatrix} E_0 \bar{S}_1^T & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\
\tilde{E}_1 &= \begin{bmatrix} E_1 \bar{Y} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \tilde{E}_2 = \begin{bmatrix} 0 & 0 & E_2 \bar{Y} & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\
S_1^{-1} &= \bar{S}_1, \bar{M}_j = \bar{S}_1 M_j \bar{S}_1^T, \bar{N}_j = \bar{S}_1 \bar{N}_j \bar{S}_1^T, j = 1, 2, \bar{P} = \bar{S}_1 P \bar{S}_1^T, \bar{Q}_i = \bar{S}_1 Q_i \bar{S}_1^T, i = 1, \ldots, 4, \\
\bar{Y} &= K \bar{S}_1^T.
\end{aligned}$$

Proof. Consider the following Lyapunov-Krasovskii functional:

$$V(t) = x^T(t)Px(t) + \sum_{i=1}^{2} \int_{t-h_i}^{t} x^T(\theta)Q_i x(\theta)d\theta + \int_{t-h}^{t} x^T(\theta)Q_3 x(\theta)d\theta$$
$$+ \int_{t-h}^{t-h_1} x^T(\theta)Q_4 x(\theta)d\theta + h_1 \int_{t-h_1}^{t}\int_{\theta}^{t} \dot{x}^T(\varphi)R_1\dot{x}(\varphi)d\varphi d\theta + \bar{h}^{-1}\int_{t-h_2}^{t-h_1}\int_{\theta}^{t} \dot{x}^T(\varphi)R_2\dot{x}(\varphi)d\varphi d\theta. \tag{5.12}$$

Differentiating (5.12) with respect to time, one obtains

$$\begin{aligned}\dot{V}(t) &= 2x^T(t)P\dot{x}(t) + \sum_{i=1}^{3} x^T(t)Q_i x(t) - \sum_{i=3}^{4} x^T(t-h)Q_i x(t-h)\\ &-x^T(t-h_1)(Q_1-Q_4)x(t-h_1) - x^T(t-h_2)Q_2x(t-h_2)\\ &+\dot{x}^T(t)(h_1^2R_1+R_2)\dot{x}(t) - h_1\int_{t-h_1}^{t} \dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta - \bar{h}^{-1}\int_{t-h_2}^{t-h_1} \dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta.\end{aligned} \tag{5.13}$$

The stability of the (5.5) can be analyzed by checking $\dot{V}(t)$ is less than zero or not. By referring Lemma 5.1, the LHS of (5.6) is added to (5.13) in order to ensure that $\dot{V}(t)$ is taken along the state trajectory of (5.5). Then, it becomes

$$\begin{aligned}&2X^T(t)\begin{bmatrix}S_1^T & S_2^T & S_3^T & S_4^T & S_5^T\end{bmatrix}^T\{-\dot{x}(t)+A_0x(t)+B_1Kx(t)+B_2Kx(t-h)\}\\ &+\sum_{k=0}^{2}\varepsilon_k X^T(t)\hat{D}_k^T\hat{D}_kX(t) + \varepsilon_0^{-1}x^T(t)E_0^TE_0x(t) + \varepsilon_1^{-1}x^T(t)K^TE_1^TE_1Kx(t)\\ &+\varepsilon_2^{-1}x^T(t-h)K^TE_2^TE_2Kx(t-h) + 2x^T(t)P\dot{x}(t) + \sum_{i=1}^{3}x^T(t)Q_ix(t)\\ &-\sum_{i=3}^{4}x^T(t-h)Q_ix(t-h) - x^T(t-h_1)(Q_1-Q_4)x(t-h_1)\\ &-x^T(t-h_2)Q_2x(t-h_2) + \dot{x}^T(t)(h_1^2R_1+R_2)\dot{x}(t)\\ &-h_1\int_{t-h_1}^{t}\dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta - \bar{h}^{-1}\int_{t-h_2}^{t-h_1}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta < 0.\end{aligned} \tag{5.14}$$

Approximating the two integral terms in the RHS of (5.14) using Lemma 1.2, (5.14) can be written as

$$X^T(t)\left\{\hat{\Theta}+\rho\Phi_1R_2^{-1}\Phi_1^T+(1-\rho)\Phi_2R_2^{-1}\Phi_2^T\right\}X(t)<0, \tag{5.15}$$

where

$$\begin{aligned}
\hat{\Theta} &= [\hat{\Theta}_{ij}]_{i,j=1,..,5}, \quad with \\
\hat{\Theta}_{11} &= Q_1+Q_2+Q_3-R_1+S_1A_0+A_0^TS_1^T+S_1B_1K+K^TB_1{}^TS_1^T \\
&\quad +\sum_{k=0}^{2}\varepsilon_kS_1D_kD_k^TS_1^T,+\varepsilon_1^{-1}E_0^TE_0+\varepsilon_2^{-1}K^TE_1^TE_1K, \\
\hat{\Theta}_{12} &= R_1+K^TB_1^TS_2^T+A_0^TS_2^T+\sum_{k=0}^{2}\varepsilon_kS_1D_kD_k^TS_2^T, \\
\hat{\Theta}_{13} &= A_0^TS_3^T+K^TB_1^TS_3^T+S_1B_2K+\sum_{k=0}^{2}\varepsilon_kS_1D_kD_k^TS_3^T \\
\hat{\Theta}_{14} &= A_0^TS_4^T+K^TB_1^TS_4^T+\sum_{k=0}^{2}\varepsilon_kS_1D_kD_k^TS_4^T \\
\hat{\Theta}_{15} &= P-S_1+A_0^TS_5^T+K^TB_1{}^TS_5^T+\sum_{k=0}^{2}\varepsilon_kS_1D_kD_k^TS_5^T, \\
\hat{\Theta}_{22} &= Q_4-Q_1-R_1+\bar{h}^{-1}(M_1+M_1^T)+\sum_{k=0}^{2}\varepsilon_kS_2D_kD_k^TS_2^T, \\
\hat{\Theta}_{23} &= \bar{h}^{-1}(-M_1+N_1^T)+S_2B_2K+\sum_{k=0}^{2}\varepsilon_kS_2D_kD_k^TS_3^T, \\
\hat{\Theta}_{24} &= \sum_{k=0}^{2}\varepsilon_kS_2D_kD_k^TS_4^T,\hat{\Theta}_{25}=-S_2+\sum_{k=0}^{2}\varepsilon_kS_2D_kD_k^TS_5^T, \\
\hat{\Theta}_{33} &= -(Q_3+Q_4)+S_3B_2K+K^TB_2^TS_3^T+\bar{h}^{-1}(-N_1-N_1^T) \\
&\quad +\bar{h}^{-1}(M_2+M_2^T)+\sum_{k=0}^{2}\varepsilon_kS_3D_kD_k^TS_3^T+\varepsilon_3^{-1}K^TE_2^TE_2K, \\
\hat{\Theta}_{34} &= \bar{h}^{-1}(-M_2+N_2^T)+K^TB_2^TS_4^T+\sum_{k=0}^{2}\varepsilon_kS_3D_kD_k^TS_4^T, \\
\hat{\Theta}_{35} &= -S_3+K^TB_2^TS_5^T+\sum_{k=0}^{2}\varepsilon_kS_3D_kD_k^TS_5^T, \\
\hat{\Theta}_{44} &= -Q_2-\bar{h}^{-1}(-N_2-N_2^T)+\sum_{k=0}^{2}\varepsilon_kS_4D_kD_k^TS_4^T,
\end{aligned}$$

$$\begin{aligned}
\hat{\Theta}_{45} &= -S_4 + \sum_{k=0}^{2} \varepsilon_k S_4 D_k D_k^T S_5^T, \\
\hat{\Theta}_{55} &= -S_5 - S_5^T + \{h_1{}^2 R_1 + R_2\} + \sum_{k=0}^{2} \varepsilon_k S_5 D_k D_k^T S_5^T, \\
\Phi_1 &= \begin{bmatrix} 0 & M_1^T & N_1^T & 0 & 0 \end{bmatrix}^T, \Phi_2 = \begin{bmatrix} 0 & 0 & M_2^T & N_2^T & 0 \end{bmatrix}^T, \rho = \frac{h - h_1}{\bar{h}}, 0 \le \rho \le 1.
\end{aligned}$$

Now, the stability requirement for (5.5) is

$$\left\{ \hat{\Theta} + \rho \Phi_1 R_2^{-1} \Phi_1^T + (1-\rho) \Phi_2 R_2^{-1} \Phi_2^T \right\} < 0. \tag{5.16}$$

The LHS of (5.16) is uncertain due to the uncertain ρ. However, it can be represented in terms of two certain vertices as:

$$\hat{\Theta} + \Phi_l R_2^{-1} \Phi_l^T < 0, \quad l = 1, 2. \tag{5.17}$$

Finally, using Schur complement on (5.17), one obtains

$$\begin{bmatrix} \hat{\Theta} & \Phi_l \\ * & -R_2 \end{bmatrix} < 0, \quad l = 1, 2. \tag{5.18}$$

To eliminate the non-linear terms in (5.18), consider S_2, S_3, S_4 and S_5 as:

$$S_2 = \lambda S_1, S_3 = \beta S_1, S_4 = \gamma S_1, S_5 = \alpha S_1,$$

and pre-multiplying and post-multiplying $diag\left\{ S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \right\}$ and its transpose, and subsequently adopting the linear change in variables as:

$$\begin{aligned}
S_1^{-1} &= \bar{S}_1, \bar{M}_j = \bar{S}_1 M_j \bar{S}_1^T, \bar{N}_j = \bar{S}_1 \bar{N}_j \bar{S}_1^T, j = 1, 2, \bar{P} = \bar{S}_1 P \bar{S}_1^T, \bar{Q}_i = \bar{S}_1 Q_i \bar{S}_1^T, i = 1, \ldots, 4, \\
\bar{Y} &= K \bar{S}_1^T.
\end{aligned}$$

One obtains

$$\check{\Theta} = \begin{bmatrix} \tilde{\Theta} & \bar{\Phi}_l \\ * & -\bar{R}_2 \end{bmatrix} < 0, \quad l = 1, 2. \tag{5.19}$$

where

$$\tilde{\Theta} = [\tilde{\Theta}_{ij}]_{i,j=1,..,6}, \quad with$$

$$
\begin{aligned}
\tilde{\Theta}_{11} &= \sum_{i=1}^{3}\bar{Q}_i - \bar{R}_1 + A_0\bar{S}_1^T + \bar{S}_1A_0^T + B_1\bar{Y} + \bar{Y}^TB_1^T + \varepsilon_0^{-1}\bar{S}_1E_0^TE_0\bar{S}_1^T + \varepsilon_2^{-1}\bar{Y}^TE_1^TE_1\bar{Y} \\
&\quad + \sum_{k=0}^{2}\varepsilon_kD_kD_k^T, \\
\tilde{\Theta}_{12} &= \bar{R}_1 + \lambda\bar{S}_1A_0^T + \lambda\bar{Y}^TB_1^T + \lambda\sum_{k=0}^{2}\varepsilon_kD_kD_k^T, \\
\tilde{\Theta}_{13} &= \beta\bar{S}_1A_0^T + \beta\bar{Y}^TB_1^T + B_2\bar{Y} + \beta\sum_{k=0}^{2}\varepsilon_kD_kD_k^T, \\
\tilde{\Theta}_{14} &= \gamma\bar{S}_1A_0^T + \gamma\bar{Y}^TB_1^T + \gamma\sum_{k=0}^{2}\varepsilon_kD_kD_k^T, \\
\tilde{\Theta}_{15} &= \bar{P} - \bar{S}_1^T + \alpha\bar{S}A_0^T + \alpha\bar{Y}^TB_1^T + \alpha\sum_{k=0}^{2}\varepsilon_kD_kD_k^T, \\
\tilde{\Theta}_{22} &= -(\bar{Q}_1 - \bar{Q}_4) - \bar{R}_1 + \bar{h}^{-1}\left[\bar{M}_1 + \bar{M}_1^T\right] + \lambda^2\sum_{k=0}^{2}\varepsilon_kD_kD_k^T, \\
\tilde{\Theta}_{23} &= \bar{h}^{-1}\left[-\bar{M}_1 + \bar{N}_1^T\right] + \lambda B_2\bar{Y} + \lambda\beta\sum_{k=0}^{2}\varepsilon_kD_kD_k^T, \\
\tilde{\Theta}_{24} &= \lambda\gamma\sum_{k=0}^{2}\varepsilon_kD_kD_k^T, \tilde{\Theta}_{25} = -\lambda\bar{S}_1^T + \lambda\alpha\sum_{k=0}^{2}\varepsilon_kD_kD_k^T, \\
\tilde{\Theta}_{33} &= -\sum_{i=3}^{4}\bar{Q}_i + \bar{h}^{-1}\left[-\bar{N}_1 - \bar{N}_1^T\right] + \bar{h}^{-1}\left[\bar{M}_2 + \bar{M}_2^T\right] + \beta B_2\bar{Y} \\
&\quad + \beta\bar{Y}^TB_2^T + \varepsilon_2^{-1}\bar{Y}^TE_2^TE_2\bar{Y} + \beta^2\sum_{k=0}^{2}\varepsilon_kD_kD_k^T, \\
\tilde{\Theta}_{34} &= \bar{h}^{-1}\left[-\bar{M}_2 + \bar{N}_2^T\right] + \gamma\bar{Y}^TB_2^T + \gamma\alpha\sum_{k=0}^{2}\varepsilon_kD_kD_k^T, \\
\tilde{\Theta}_{35} &= -\beta\bar{S}_1^T + \alpha\bar{Y}^TB_2^T + \beta\alpha\sum_{k=0}^{2}\varepsilon_kD_kD_k^T, \\
\tilde{\Theta}_{44} &= -\bar{Q}_2 + \bar{h}^{-1}\left[-\bar{N}_2 - \bar{N}_2^T\right] + \gamma^2\sum_{k=0}^{2}\varepsilon_kD_kD_k^T, \\
\tilde{\Theta}_{45} &= -\gamma S_1^T + \gamma\alpha\sum_{k=0}^{2}\varepsilon_kD_kD_k^T, \\
\tilde{\Theta}_{55} &= h_1^2\bar{R}_1 + \bar{R}_2 - \alpha\bar{S}_1 - \alpha\bar{S}_1^T + \alpha^2\sum_{k=0}^{2}\varepsilon_kD_kD_k^T.
\end{aligned}
$$

Separating the nonlinear terms in $\check{\Theta}$, one can write

$$\Theta_k + \Xi_0 + \Xi_1 + \Xi_2 < 0, \tag{5.20}$$

where

$$\begin{aligned}
\Xi_0 &= \begin{bmatrix} \varepsilon_0^{-1}\bar{S}_1 E_0^T E_0 \bar{S}_1^T & 0_{5\times5} \\ 0_{5\times1} & 0_{1\times5} \end{bmatrix}, \Xi_1 = \begin{bmatrix} \varepsilon_1^{-1}\bar{Y}_1^T E_1^T E_1 \bar{Y}_1 & 0_{5\times5} \\ 0_{5\times1} & 0_{1\times5} \end{bmatrix}, \\
\Xi_2 &= \begin{bmatrix} \Xi_{21} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix}, \Xi_{21} = \begin{bmatrix} 0_{2\times2} & 0_{2\times1} \\ 0_{1\times2} & \varepsilon_2^{-1}\bar{Y}_1^T E_2^T E_2 \bar{Y}_1 \end{bmatrix}.
\end{aligned}$$

Finally, employing Schur Complement thrice on (5.20), one obtains (5.11). □

The effectiveness of the derived criterion is validated with numerical examples presented in the next subsection.

5.3.2 Numerical examples

The less conservativeness of the resulting criterion compared to some existing results in Theorem 5.1 is verified through two numerical examples in this section. For the purpose, the tuning parameters λ, β, γ and α are searched using *fminsearch* program of $MATLAB^{®}$.

Example 5.1. Consider a system of (5.1) with [14]

$$\begin{aligned}
\dot{x}(t) &= (A_0 + \Delta A_0)x(t) + B_2 u(t-h),\ t \geq 0, \\
x(0) &= x_{0,} u(t) = \phi(t),\ t \in [-0.2, 0],
\end{aligned} \tag{5.21}$$

where

$$A_0 = \begin{bmatrix} 0 & 1 \\ -1.25 & -3 \end{bmatrix}, \quad \Delta A_0 = \begin{bmatrix} 0 & 0 \\ q & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad h_1 = 0, \quad h_2 = 0.2.$$

In the above system description, q is an uncertain parameter with bounding $|q| \leq \eta$. The conservativeness of the proposed theory on stabilization is verified in terms of designing a controller that maximizes tolerable η. The maximum η is obtained using Theorem 5.1 as 10.0048. The corresponding tuning parameters λ, β, γ and α are tuned at 4.9119, 0.1613, -1.0198 and 1.0465 respectively corresponding to a controller $K = [-13.9272 \quad -3.6382]$. In [14, 174], the stabilizing controllers are of the form $u(t) = Kz(t)$, where $z(t) = x(t) +$

$\int_{t-h_0}^{t} e^{A(t-s-h_0)}B_1u(s)ds$. Such a controller is complicated in structure due to its integral part and is difficult to implement. But the approach in this chapter considers a static state feedback controller of the form $u(t) = Kx(t)$, which is simple in structure and easy to implement. A clear picture of comparison of present result with some existing ones is presented in Table 5.1.

The Theorem 5.1 provides a less conservative robust stabilization criterion for systems with input-delay than that of existing results. The reasons of less conservativeness of the Theorem 5.1 are stated as follows: (i) A special type of LK functional i.e (5.12) is considered to describe the energy functional of the system, (ii) The conventional way to introduce the state information is by replacing the $\dot{x}(t)$ term directly from the state equation in the derivative of the LK functional i.e. (5.13). However, such replacement in stabilization problems of time-delay system does not yield a convenient LMI form for stabilization criterion. Alternatively, one may convert the state equation (5.5) suitably into a quadratic form and thereby appending the same to $\dot{V}_i$ term so that the replacement of $\dot{x}$ can be avoided. Here, we have used a quadratic form introducing five free-matrix variables such as S_1, S_2, S_3, S_4 and S_5 in order to explore the interplay of the different signals in the state dynamics, (iii) Jensen's inequality is used to approximate the integral terms in the derivative of the LK functional.

To validate the control design using Theorem 5.1, the simulation result with initial condition $x(t) = [-1, 3]$, $t \in [-0.2, 0]$ is presented in Fig. 5.1. The result shows that all the states of the system are stable using the designed controller ($K = [-13.9272 \quad -3.6382]$) by Theorem 5.1.

Table 5.1: Comparison of robustness *maximum* η

Approach	maximum η	Structure of $u(t)$
[174]	7.2568	$u(t) = Kz(t)$, where $z(t) = x(t) + \int_{t-h_0}^{t} e^{A(t-s-h_0)}B_1u(s)ds$
[14]	10.8485	$u(t) = Kz(t)$, where $z(t) = x(t) + \int_{t-h_0}^{t} e^{A(t-s-h_0)}B_1u(s)ds$
Theorem 5.1	10.0048	$u(t) = Kx(t)$ where $K = [-13.9272 \quad -3.6382]$

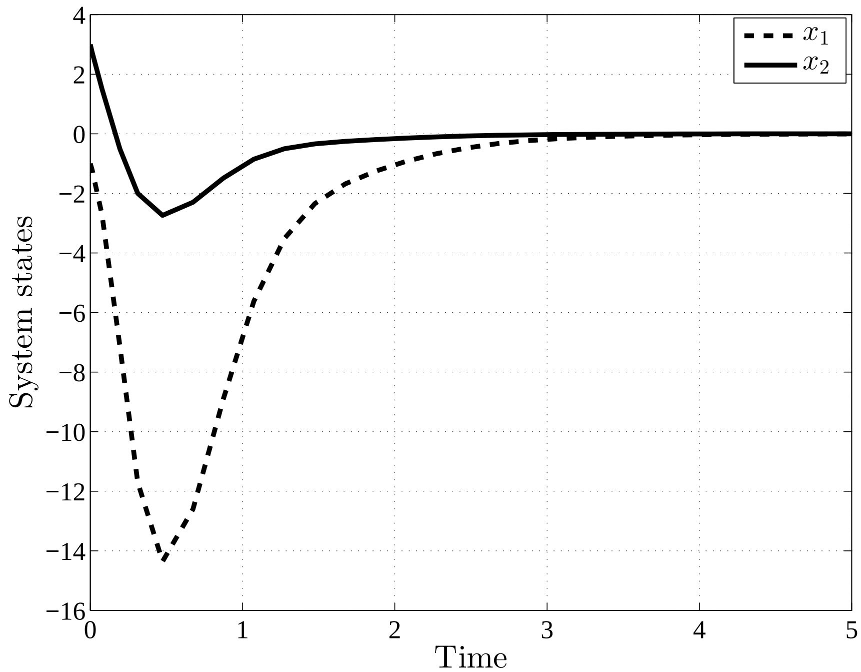

Figure 5.1: Variation of system states with respect to time for Example 5.1

Example 5.2. Next, consider system (5.1) with [108]

$$\begin{aligned} A_0 &= \begin{bmatrix} -0.08 & -0.03 & 0.2 \\ 0.2 & -0.04 & -0.005 \\ -0.06 & 0.2 & -0.07 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} -0.1 \\ -0.2 \\ 0.1 \end{bmatrix}, D_0 = D_1 = D_2 = 0, \\ E_0 &= E_1 = E_2 = 0. \end{aligned}$$

For this system, using Theorem 5.1, the maximum delay margin ($\bar{h}$) is obtained to be 9.1626. The corresponding tuning parameters λ, β, γ and α are tuned at 0.1408, 0.0121, 0.0056 and 475.3271 respectively that yields a controller $K = [0.4464 \quad 0.5653 \quad 0.4855]$. A comparison of the present result with existing ones is presented in Table 5.2 that shows the less conservativeness of the developed criterion. The designed controller using Theorem 5.1 is used to obtain the simulation result with initial condition $x(t) = [-1, 3, -2]$, $t \in [-9.1626, 0]$ shown in Fig. 5.2. The simulation result (norm of the states of the system) shows that the system is stable at $\bar{h} = 9.1626$.

Table 5.2: Maximum Tolerable Delay Bound ($\bar{h}$)

Approach	$\bar{h}$
[108]	5
Theorem 5.1	9.1626

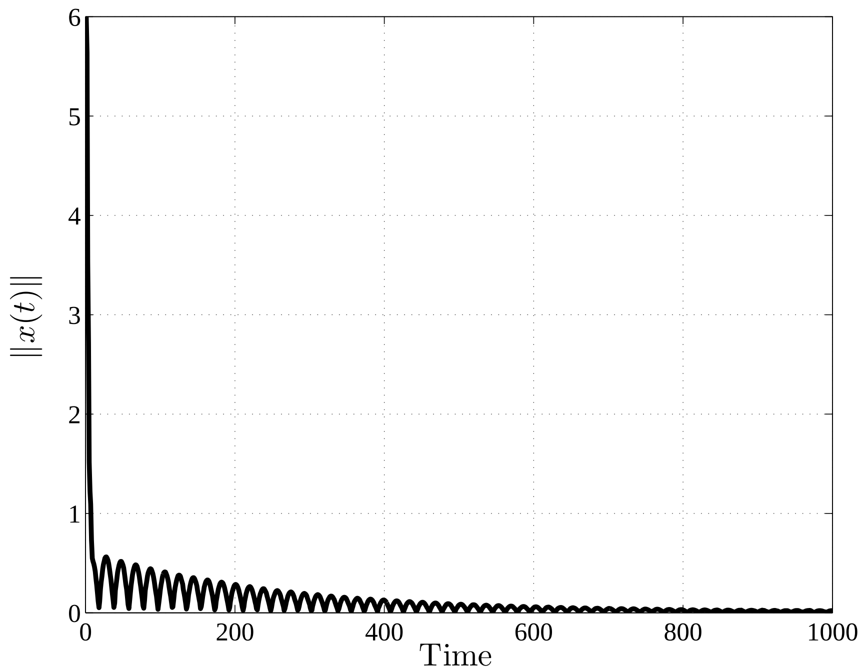

Figure 5.2: Variation of norm of the state vector with respect to time for Example 5.2

5.4 Stabilization using delay-decomposition

For further reduction of the conservatism, the decomposition approach is used to design robust controller for (5.1). To derive such criterion, a simple static state feedback controller of the form (5.4) is used.

5.4.1 Stabilization criterion

The following theorem presents an LMI based controller design for system (5.1).

Theorem 5.2. *System (5.5) is stable if there exist matrices* $\bar{P} > 0$, $\bar{Q}_j > 0$, $j = 1, \ldots, 4$,

$\bar{R}_i > 0$ *and arbitrary matrices* $\bar{S}_l$, $\bar{M}_i$, $\bar{N}_i$, $l = 1 \ldots 5$, $i = 1, 2$, *that satisfy the following LMI:*

$$\begin{bmatrix} \Upsilon_k & \check{E}_0 & \check{E}_1 & \check{E}_2 \\ * & -\varepsilon_0 I & 0 & 0 \\ * & * & -\varepsilon_1 I & 0 \\ * & * & * & -\varepsilon_2 I \end{bmatrix} < 0, \tag{5.22}$$

where

$$\begin{aligned}
\Upsilon_k &= \begin{bmatrix} \bar{\Upsilon} & \delta\bar{\phi}_k \\ * & -\bar{R}_2 \end{bmatrix}, \quad k = 1,2, \quad \bar{\Upsilon} = \Upsilon + \bar{D}, \quad \bar{D} = \sum_{k=1}^{3} \varepsilon_k \bar{D}_k^T \bar{D}_k, \\
\bar{D}_k &= \begin{bmatrix} D_k^T & \lambda D_k^T & \beta D_k^T & \gamma D_k^T & \alpha D_k^T \end{bmatrix}, \\
\bar{\phi}_1 &= \begin{bmatrix} 0 & \bar{M}_1^T & \bar{N}_1^T & 0 & 0 \end{bmatrix}^T, \bar{\phi}_2 = \begin{bmatrix} 0 & 0 & \bar{M}_2^T & \bar{N}_2^T & 0 \end{bmatrix}^T, \Upsilon = [\Upsilon_{ij}]_{i,j=1,\ldots,5}, \\
\Upsilon_{11} &= \sum_{i=1}^{3} \bar{Q}_i - \bar{R}_1 + A_0\bar{S}_1^T + \bar{S}_1 A_0^T + B_1\bar{Y} + \bar{Y}^T B_1^T, \Upsilon_{12} = \bar{R}_1 + \lambda\bar{S}_1 A_0^T + \lambda\bar{Y}^T B_1^T, \\
\Upsilon_{13} &= \beta\bar{S}_1 A_0^T + \beta\bar{Y}^T B_1^T + B_2\bar{Y}, \Upsilon_{14} = \gamma\bar{S}_1 A_0^T + \gamma\bar{Y}^T B_1^T, \\
\Upsilon_{15} &= \bar{P} - \bar{S}_1^T + \alpha\bar{S}A_0^T + \alpha\bar{Y}^T B_1^T, \Upsilon_{22} = -(\bar{Q}_2 - \bar{Q}_4) - \bar{R}_1 + \delta\left[\bar{M}_1 + \bar{M}_1^T\right], \\
\Upsilon_{23} &= \delta\left[-\bar{M}_1 + \bar{N}_1^T\right] + \lambda B_2\bar{Y}, \Upsilon_{24} = 0, \Upsilon_{25} = -\lambda\bar{S}_1^T, \\
\Upsilon_{33} &= -\sum_{i=3}^{4} \bar{Q}_i + \delta\left[-\bar{N}_1 - \bar{N}_1^T\right] + \delta\left[\bar{M}_2 + \bar{M}_2^T\right] + \beta B_2\bar{Y} + \beta\bar{Y}^T B_2^T, \\
\Upsilon_{34} &= \delta\left[-\bar{M}_2 + \bar{N}_2^T\right] + \gamma\bar{Y}^T B_2^T, \Upsilon_{35} = -\beta\bar{S}_1^T + \alpha\bar{Y}^T B_2^T, \\
\Upsilon_{44} &= -\bar{Q}_1 + \delta\left[-\bar{N}_2 - \bar{N}_2^T\right], \Upsilon_{45} = -\gamma\bar{S}_1^T, \\
\Upsilon_{55} &= (\bar{h} - \delta)^2\bar{R}_1 + \delta^2\bar{R}_2 - \alpha\bar{S}_1 - \alpha\bar{S}_1^T, \check{E}_0 = \begin{bmatrix} E_0\bar{S}_1^T & \boldsymbol{0}_{1\times 6} \end{bmatrix}^T, \check{E}_1 = \begin{bmatrix} E_1\bar{Y} & \boldsymbol{0}_{1\times 6} \end{bmatrix}^T, \\
\check{E}_2 &= \begin{bmatrix} 0 & 0 & E_2\bar{Y} & \boldsymbol{0}_{1\times 4} \end{bmatrix}^T, \bar{S}_1 = S_1^{-1}, \bar{P} = \bar{S}_1 P \bar{S}_1^T, \bar{M}_i = \bar{S}_1 M_i \bar{S}_1^T, \\
\bar{N}_i &= \bar{S}_1 N_i \bar{S}_1^T, \quad i = 1,2, \bar{Q}_j = \bar{S}_1 Q_j \bar{S}_1^T, \quad j = 1,..4, \bar{Y} = K\bar{S}_1^T.
\end{aligned}$$

Proof. Consider a simple LK functional for i^{th} interval that $h \in [h_{(i-1)}, h_i]$ as:

$$\begin{aligned} V_i(x_t, \dot{x}_t) &= x^T(t)Px(t) + \sum_{j=1}^{2} \int_{t-h_{(i+1-j)}}^{t} x^T(\theta)Q_j x(\theta)d\theta + \int_{t-h}^{t} x^T(\theta)Q_3 x(\theta)d\theta \\ &+ \int_{t-h}^{t-h_{(i-1)}} x^T(\theta)Q_4 x(\theta)d\theta + h_{(i-1)} \int_{t-h_{(i-1)}}^{t} \int_{\theta}^{t} \dot{x}^T(\phi)R_1\dot{x}(\phi)d\phi d\theta + \delta \int_{t-h_i}^{t-h_{(i-1)}} \int_{\theta}^{t} \dot{x}^T(\phi)R_2\dot{x}(\phi)d\phi d\theta. \end{aligned} \tag{5.23}$$

Differentiating V_i with respect to time along the state trajectory of (5.5) yields

$$\begin{aligned} \dot{V}_i(x_t, \dot{x}_t) &= 2x^T(t)P\dot{x}(t) + \sum_{k=1}^{3} x^T(t)Q_k x(t) - x^T(t-h_{(i-1)})(Q_2 - Q_4)x(t-h_{(i-1)}) \\ &- \sum_{k=3}^{4} x^T(t-h)Q_k x(t-h) - x^T(t-h_i)Q_1 x(t-h_i) + \dot{x}^T(t)\left\{h_{(i-1)}^2 R_1 + \delta^2 R_2\right\}\dot{x}(t) \\ &-h_{(i-1)} \int_{t-h_{(i-1)}}^{t} \dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta - \delta \int_{t-h_i}^{t-h_{(i-1)}} \dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta. \end{aligned} \tag{5.24}$$

Instead of replacing $\dot{x}(t)$ by directly using (5.5), we consider the quadratic formulation of the system dynamics (5.5) as:

$$\begin{aligned} &2\left\{x^T(t)S_1 + x^T(t-h_{i-1})S_2 + x^T(t-h)S_3 + x^T(t-h_i)S_4 + \dot{x}^T(t)S_5\right\} \\ &\quad\times\left\{-\dot{x}(t) + A_0(t)x(t) + B_1(t)Kx(t) + B_2(t)Kx(t-h)\right\} = 0, \end{aligned} \tag{5.25}$$

where $S_k, k = 1, \ldots, 5$ are arbitrary matrices of appropriate dimensions. Next, the bounds of the uncertain terms in (5.25) are obtained. Following Lemma 5.1, one may write

$$\begin{aligned} &2\xi^T(t)\begin{bmatrix} S_1^T & S_2^T & S_3^T & S_4^T & S_5^T \end{bmatrix}^T \begin{bmatrix} D_0F(t)E_0x(t) & D_1F(t)E_1Kx(t) & D_2F(t)E_2Kx(t-h) \end{bmatrix} \\ &\quad\le \sum_{k=1}^{3} \varepsilon_k \xi^T(t)\hat{D}_k^T\hat{D}_k\xi(t) + \varepsilon_0^{-1}x^T(t)E_0^T E_0 x(t) + \varepsilon_1^{-1}x^T(t)K^T E_1^T E_1 Kx(t) \\ &\qquad + \varepsilon_2^{-1}x^T(t-h)K^T E_2^T E_2 Kx(t-h), \end{aligned} \tag{5.26}$$

where

$$
\begin{aligned}
\xi(t) &= \begin{bmatrix} x^T(t) & x^T(t-h_{(i-1)}) & x^T(t-h) & x^T(t-h_i) & \dot{x}^T(t) \end{bmatrix}^T, \\
\hat{D}_k &= \begin{bmatrix} D_k^T S_1^T & D_k^T S_2^T & D_k^T S_3^T & D_k^T S_4^T & D_k^T S_5^T \end{bmatrix}.
\end{aligned}
$$

Adding (5.25) with (5.24) by approximating the uncertain terms using (5.26), one obtains

$$
\begin{aligned}
\dot{V}(t) \leq{} & 2\xi^T(t)\begin{bmatrix} S_1^T & S_2^T & S_3^T & S_4^T & S_5^T \end{bmatrix}^T \{-\dot{x}(t) + (A_0 + B_1 K)x(t) + B_2 x(t-h)\} \\
& + \sum_{k=1}^{3} \varepsilon_{k-1}\xi^T(t)\hat{D}_{k-1}^T \hat{D}_{k-1}\xi(t) + \varepsilon_0^{-1} x^T(t) E_0^T E_0 x(t) + \varepsilon_1^{-1} x^T(t) K^T E_1^T E_1 K x(t) \\
& + \varepsilon_2^{-1} x^T(t-h) K^T E_2^T E_2 K x(t-h) + 2x^T(t) P\dot{x}(t) + \sum_{k=1}^{3} x^T(t) Q_k x(t) \\
& - x^T(t-h_{(i-1)})(Q_2 - Q_4)x(t-h_{(i-1)}) - \sum_{k=3}^{4} x^T(t-h) Q_k x(t-h) - x^T(t-h_i) Q_1 x(t-h_i) \\
& + \dot{x}^T(t)\left\{h_{(i-1)}^2 R_1 + \delta^2 R_2\right\}\dot{x}(t) - h_{(i-1)} \int\limits_{t-h_{(i-1)}}^{t} \dot{x}^T(\theta) R_1 \dot{x}(\theta) d\theta - \delta \int\limits_{t-h_i}^{t-h_{(i-1)}} \dot{x}^T(\theta) R_2 \dot{x}(\theta) d\theta.
\end{aligned}
\tag{5.27}
$$

Following Lemma 1.2, the first integral term in (5.27) satisfies

$$
- h_{(i-1)} \int\limits_{t-h_{(i-1)}}^{t} \dot{x}^T(\theta) R_1 \dot{x}(\theta) d\theta \leq \begin{bmatrix} x(t) \\ x(t-h_{(i-1)}) \end{bmatrix} \begin{bmatrix} -R_1 & R_1 \\ * & -R_1 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h_{(i-1)}) \end{bmatrix}. \tag{5.28}
$$

and the second one due to having uncertain delay parameter satisfies

$$
\begin{aligned}
-\delta \int\limits_{t-h_i}^{t-h_{(i-1)}} \dot{x}^T(\theta) R_2 \dot{x}(\theta) d\theta = \quad & \begin{bmatrix} x(t-h_{(i-1)}) \\ x(t-h) \end{bmatrix}^T \left\{ \delta \begin{bmatrix} M_1 + M_1^T & -M_1 + N_1^T \\ * & -N_1 - N_1^T \end{bmatrix} \right. \\
& \left. + \delta^2 \rho \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} R_2^{-1} \begin{bmatrix} M_1 \\ N_1 \end{bmatrix}^T \right\} \begin{bmatrix} x(t-h_{(i-1)}) \\ x(t-h) \end{bmatrix} \\
& + \begin{bmatrix} x(t-h) \\ x(t-h_{(i)}) \end{bmatrix}^T \left\{ \delta \begin{bmatrix} M_2 + M_2^T & -M_2 + N_2^T \\ * & -N_2 - N_2^T \end{bmatrix} \right.
\end{aligned}
$$

$$+\delta^2(1-\rho)\begin{bmatrix} M_2 \\ N_2 \end{bmatrix} R_2^{-1} \begin{bmatrix} M_2 \\ N_2 \end{bmatrix}^T \Bigg\} \begin{bmatrix} x(t-h) \\ x(t-h_{(i)}) \end{bmatrix}. \quad (5.29)$$

Approximating the integral terms of (5.27) using (5.28) and (5.29), one obtains

$$\dot{V}_i(t) \le \xi^T(t)(\psi + h_{i-1}^2\Omega_i + \rho\delta^2\phi_1 R_2^{-1}\phi_1^T + (1-\rho)\delta^2\phi_2 R_2^{-1}\phi_2^T)\xi(t), \quad (5.30)$$

where

$$\begin{aligned}
\psi_{11} &= \sum_{i=1}^{3} Q_i - R_1 + S_1 A_0 + A_0^T S_1^T + S_1 B_1 K + K^T B_1^T S_1^T + \varepsilon_0^{-1} E_0^T E_0 \\
&\quad + \varepsilon_1^{-1} K^T E_1^T E_1 K + \sum_{k=0}^{2} \varepsilon_k S_1 D_k D_k^T S_1^T, \\
\psi_{12} &= R_1 + A_0^T S_2^T + K^T B_1^T S_2^T + \sum_{k=0}^{2} \varepsilon_k S_1 D_k D_k^T S_2^T, \\
\psi_{13} &= A_0^T S_3^T + K^T B_1^T S_3^T + S_1 B_2 K + \sum_{k=0}^{2} \varepsilon_k S_1 D_k D_k^T S_3^T, \\
\psi_{14} &= A_0^T S_4^T + K^T B_1^T S_4^T + \sum_{k=0}^{2} \varepsilon_k S_1 D_k D_k^T S_4^T, \\
\psi_{15} &= P - S_1 + A_0^T S_5^T + K^T B_1^T S_5^T + \sum_{k=0}^{2} \varepsilon_k S_1 D_k D_k^T S_5^T, \\
\psi_{22} &= -(Q_2 - Q_4) - R_1 + \delta\left[M_1 + M_1^T\right] + \sum_{k=0}^{2} \varepsilon_k S_2 D_k D_k^T S_2^T, \\
\psi_{23} &= \delta\left[-M_1 + N_1^T\right] + S_2 B_2 K + \sum_{k=0}^{2} \varepsilon_k S_2 D_k D_k^T S_3^T, \\
\psi_{24} &= \sum_{k=0}^{2} \varepsilon_k S_2 D_k D_k^T S_4^T, \psi_{25} = -S_2 + \sum_{k=0}^{2} \varepsilon_k S_2 D_k D_k^T S_5^T, \\
\psi_{33} &= -\sum_{i=3}^{4} Q_i + \delta\left[-N_1 - N_1^T\right] + \delta\left[M_2 + M_2^T\right] + S_3 B_2 K + K^T B_2^T S_3^T \\
&\quad + \varepsilon_2^{-1} K^T E_2^T E_2 K + \sum_{k=0}^{2} \varepsilon_k S_3 D_k D_k^T S_3^T, \\
\psi_{34} &= \delta\left[-M_2 + N_2^T\right] + K^T B_2^T S_4^T + \sum_{k=0}^{2} \varepsilon_k S_3 D_{k-1} D_{k-1}^T S_4^T,
\end{aligned}$$

$$\begin{aligned}
\psi_{35} &= -S_3 + K^T B_2^T S_5^T + \sum_{k=0}^{2} \varepsilon_k S_3 D_k D_k^T S_5^T, \\
\psi_{44} &= -Q_1 + \delta\left[-N_2 - N_2^T\right] + \sum_{k=0}^{2} \varepsilon_k S_4 D_k D_k^T S_4^T, \\
\psi_{45} &= -S_4 + \sum_{k=0}^{2} \varepsilon_k S_4 D_k D_k^T S_5^T, \psi_{55} = \delta^2 R_2 - S_5 - S_5^T + \sum_{k=0}^{2} \varepsilon_k S_4 D_k D_k^T S_5^T, \\
\rho &= \frac{\bar{h} - h_{i-1}}{\delta}, 0 \leq \rho \leq 1, \Omega_i = \begin{bmatrix} 0_{4n\times 4n} & 0_{4n\times n} \\ 0_{n\times 4n} & R_1 \end{bmatrix}.
\end{aligned}$$

Therefore, the stability requirement for the i^{th} interval is

$$\psi + h_{(i-1)}^2 \Omega_i + \delta^2 \phi_j R_2^{-1} \phi_j^T < 0, \quad j = 1, 2. \tag{5.31}$$

To this end, note that, $\Omega_i \geq 0$ and $h_{(i-1)}^2 \Omega_i$ term is maximum when $h \in [h_{(N-1)}, \bar{h}]$, the N^{th} interval. Therefore, irrespective of h lies in any of the intervals, the following condition always ensures stability of (5.5):

$$\psi + h_{(N-1)}^2 \Omega_N + \delta^2 \phi_j R_2^{-1} \phi_j^T < 0, \quad j = 1, 2. \tag{5.32}$$

One can write (5.32) in LMI form as:

$$\begin{bmatrix} \bar{\psi} & \delta\phi_k \\ * & -R_2 \end{bmatrix} < 0, \quad k = 1, 2, \tag{5.33}$$

where $\bar{\psi} = \psi + h_{(i-1)}^2 \Omega_i$.

For linearization, considering S_2, S_3, S_4, S_5 as: $S_2 = \lambda S_1$, $S_3 = \beta S_1$, $S_4 = \gamma S_1$, $S_5 = \alpha S_1$ and then, pre- and post-multiplying by $diag\left\{S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \quad S_1^{-1}\right\}$ and its transpose respectively, and subsequently adopting the change of variables.

$$\begin{aligned}
\bar{S}_1 &= S_1^{-1}, \bar{P} = \bar{S}_1 P \bar{S}_1^T, \bar{M}_i = \bar{S}_1 M_i \bar{S}_1^T, \bar{N}_i = \bar{S}_1 N_i \bar{S}_1^T, i = 1, 2, \bar{Q}_j = \bar{S}_1 Q_j \bar{S}_1^T, j = 1, \ldots, 4, \\
\bar{Y} &= K \bar{S}_1^T.
\end{aligned}$$

One obtains from (5.33),

$$\check{\psi} = \begin{bmatrix} \tilde{\psi} & \bar{\phi}_l \\ * & -\bar{R}_2 \end{bmatrix} < 0, \quad l = 1, 2, \tag{5.34}$$

where

$$
\begin{aligned}
\tilde{\psi} &= [\tilde{\psi}_{ij}]_{i,j=1,\ldots,5}, \\
\tilde{\psi}_{11} &= \sum_{i=1}^{3} \bar{Q}_i - \bar{R}_1 + A_0 \bar{S}_1^T + \bar{S}_1 A_0^T + B_1 \bar{Y} + \bar{Y}^T B_1^T + \varepsilon_1^{-1} \bar{S}_1 E_0^T E_0 \bar{S}_1^T \\
&\quad + \varepsilon_2^{-1} \bar{Y}^T E_1^T E_1 \bar{Y} + \sum_{k=0}^{2} \varepsilon_k D_k D_k^T, \\
\tilde{\psi}_{12} &= \bar{R}_1 + \lambda \bar{S}_1 A_0^T + \lambda \bar{Y}^T B_1^T + \lambda \sum_{k=0}^{2} \varepsilon_k D_k D_k^T, \\
\tilde{\psi}_{13} &= \beta \bar{S}_1 A_0^T + \beta \bar{Y}^T B_1^T + B_2 \bar{Y} + \beta \sum_{k=0}^{2} \varepsilon_k D_k D_k^T, \\
\tilde{\psi}_{14} &= \gamma \bar{S}_1 A_0^T + \gamma \bar{Y}^T B_1^T + \gamma \sum_{k=0}^{2} \varepsilon_k D_k D_k^T, \\
\tilde{\psi}_{15} &= \bar{P} - \bar{S}_1^T + \alpha \bar{S} A_0^T + \alpha \bar{Y}^T B_1^T + \alpha \sum_{k=0}^{2} \varepsilon_k D_k D_k^T, \\
\tilde{\psi}_{22} &= -(\bar{Q}_2 - \bar{Q}_4) - \bar{R}_1 + \delta \left[\bar{M}_1 + \bar{M}_1^T \right] + \lambda^2 \sum_{k=0}^{2} \varepsilon_k D_k D_k^T, \\
\tilde{\psi}_{23} &= \delta \left[-\bar{M}_1 + \bar{N}_1^T \right] + \lambda B_2 \bar{Y} + \lambda \beta \sum_{k=0}^{2} \varepsilon_k D_k D_k^T, \\
\tilde{\psi}_{24} &= \lambda \gamma \sum_{k=0}^{2} \varepsilon_k D_k D_k^T, \tilde{\psi}_{25} = -\lambda \bar{S}_1^T + \lambda \alpha \sum_{k=0}^{2} \varepsilon_k D_k D_k^T, \\
\tilde{\psi}_{33} &= -\sum_{i=3}^{4} \bar{Q}_i + \delta \left[-\bar{N}_1 - \bar{N}_1^T \right] + \delta \left[\bar{M}_2 + \bar{M}_2^T \right] + \beta B_2 \bar{Y} + \beta \bar{Y}^T B_2^T \\
&\quad + \varepsilon_3^{-1} \bar{Y}^T E_2^T E_2 \bar{Y} + \beta^2 \sum_{k=0}^{2} \varepsilon_k D_k D_k^T, \\
\tilde{\psi}_{34} &= \delta \left[-\bar{M}_2 + \bar{N}_2^T \right] + \gamma \bar{Y}^T B_2^T + \gamma \alpha \sum_{k=0}^{2} \varepsilon_k D_k D_k^T, \\
\tilde{\psi}_{35} &= -\beta \bar{S}_1^T + \alpha \bar{Y}^T B_2^T + \beta \alpha \sum_{k=0}^{2} \varepsilon_k D_k D_k^T, \\
\tilde{\psi}_{44} &= -\bar{Q}_1 + \delta \left[-\bar{N}_2 - \bar{N}_2^T \right] + \gamma^2 \sum_{k=0}^{2} \varepsilon_k D_k D_k^T,
\end{aligned}
$$

$$\tilde{\psi}_{45} = -\gamma\bar{S}_1^T + \gamma\alpha\sum_{k=0}^{2}\varepsilon_k D_k D_k^T, \tilde{\psi}_{55} = \delta^2\bar{R}_2 - \alpha\bar{S}_1 - \alpha\bar{S}_1^T + \alpha^2\sum_{k=0}^{2}\varepsilon_k D_k D_k^T.$$

Separating the nonlinear terms in $\check{\psi}$, one can write (5.34) as:

$$\Upsilon_k + \Xi_0 + \Xi_1 + \Xi_2 < 0, \tag{5.35}$$

where

$$\begin{aligned}
\Xi_0 &= \begin{bmatrix} \varepsilon_0^{-1}\bar{S}_1 E_0^T E_0 \bar{S}_1^T & 0_{5\times5} \\ 0_{5\times1} & 0_{1\times5} \end{bmatrix}, \Xi_1 = \begin{bmatrix} \varepsilon_1^{-1}\bar{Y}_1^T E_1^T E_1 \bar{Y}_1 & 0_{5\times5} \\ 0_{5\times1} & 0_{1\times5} \end{bmatrix}, \\
\Xi_2 &= \begin{bmatrix} \Xi_{21} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix}, \Xi_{21} = \begin{bmatrix} 0_{2\times2} & 0_{2\times1} \\ 0_{1\times2} & \varepsilon_2^{-1}\bar{Y}_1^T E_2^T E_2 \bar{Y}_1 \end{bmatrix}.
\end{aligned}$$

Applying Schur complement on (5.35), one obtains (5.22). □

The following corollary is a simplified form of Theorem 5.2 by eliminating the number of free matrix variables. However, this involves approximation for simplification.

Corollary 5.1. *System (5.5) is stable if there exist $\bar{P} > 0$, $\bar{Q}_k > 0$, $\bar{R}_j > 0$, $k = 1, \ldots, 4$, $j = 1, 2$ satisfying the following LMI condition:*

$$\begin{bmatrix} \tilde{\Sigma} & \check{E}_0 & \check{E}_1 & \check{E}_2 \\ * & -\varepsilon_0 I & 0 & 0 \\ * & * & -\varepsilon_1 I & 0 \\ * & * & * & -\varepsilon_2 I \end{bmatrix} < 0, \tag{5.36}$$

where

$$\begin{aligned}
\tilde{\Sigma} &= \bar{\Sigma} + \bar{D}, \bar{\Sigma} = [\bar{\Sigma}_{ij}]_{i,j=1,\ldots,5}, \bar{\Sigma}_{11} = \Upsilon_{11}, \bar{\Sigma}_{12} = \Upsilon_{12}, \bar{\Sigma}_{13} = \Upsilon_{13}, \bar{\Sigma}_{14} = \Upsilon_{14}, \\
\bar{\Sigma}_{15} &= \Upsilon_{15}, \bar{\Sigma}_{22} = \bar{Q}_4 - \bar{Q}_2 - \bar{R}_1 - \bar{R}_2, \bar{\Sigma}_{23} = \lambda B_2\bar{Y} + \bar{R}_2, \bar{\Sigma}_{24} = \Upsilon_{24}, \bar{\Sigma}_{25} = \Upsilon_{25}, \\
\bar{\Sigma}_{33} &= -\sum_{i=3}^{4}\bar{Q}_i + \beta B_2\bar{Y} + \beta\bar{Y}^T B_2^T - 2\bar{R}_2, \bar{\Sigma}_{34} = \gamma\bar{Y}^T B_2^T + \bar{R}_2, \bar{\Sigma}_{35} = \Upsilon_{35}, \\
\bar{\Sigma}_{44} &= -\bar{Q}_1 - \bar{R}_2, \bar{\Sigma}_{45} = \Upsilon_{45}, \bar{\Sigma}_{55} = \Upsilon_{55}.
\end{aligned}$$

Proof. Since the last term in (5.31) is positive definite, one may reduce the stability condition in the form of a single matrix inequalities as:

$$\psi + h_{N-1}^2\Omega_N + \delta^2\phi_1 R_2^{-1}\phi_1^T + \delta^2\phi_2 R_2^{-1}\phi_2^T < 0. \tag{5.37}$$

one may write (5.37) as:

$$\bar{\psi} + \delta^2\phi_1 R_2^{-1}\phi_1^T + \delta^2\phi_2 R_2^{-1}\phi_2^T < 0, \tag{5.38}$$

where $\bar{\psi} = \psi + h_{(N-1)}^2\Omega_N$. Separating the M_1, N_1, M_2 and N_2 terms from $\bar{\psi}$, one obtains

$$\upsilon + (\delta\phi_1)I_1^T + I_1(\delta\phi_1)^T + (\delta\phi_1)R_2^{-1}(\delta\phi_1)^T + (\delta\phi_2)I_2^T + I_2(\delta\phi_2)^T + (\delta\phi_2)R_2^{-1}(\delta\phi_2)^T < 0, \tag{5.39}$$

where

$$\begin{aligned}
\upsilon &= [\upsilon_{ij}]_{i,j=1,\dots,5}, \upsilon_{11} = \psi_{11}, \upsilon_{12} = \psi_{12}, \upsilon_{13} = \psi_{13}, \upsilon_{14} = \psi_{14}, \upsilon_{15} = \psi_{15},\\
\upsilon_{22} &= -(Q_2 - Q_4) - R_1 + \sum_{k=0}^{2}\varepsilon_k S_2 D_k D_k^T S_2^T, \upsilon_{23} = S_2 B_2 K + \sum_{k=0}^{2}\varepsilon_k S_2 D_k D_k^T S_3^T, \upsilon_{24} = \psi_{24},\\
\upsilon_{25} &= \psi_{25}, \upsilon_{33} = -\sum_{k=3}^{4} Q_k + S_3 B_2 K + K^T B_2^T S_3^T + \varepsilon_2^{-1} K^T E_2^T E_2 K + \sum_{k=0}^{2}\varepsilon_k S_3 D_k D_k^T S_3^T,\\
\upsilon_{34} &= K^T B_2^T S_4^T + \sum_{k=0}^{2}\varepsilon_k S_3 D_k D_k^T S_4^T, \upsilon_{35} = \psi_{35}, \upsilon_{44} = -Q_1 + \sum_{k=0}^{3}\varepsilon_k S_4 D_k D_k^T S_4^T, \upsilon_{45} = \psi_{45},\\
\upsilon_{55} &= \psi_{55}, I_1 = \begin{bmatrix} 0 & I & -I & 0 & 0 \end{bmatrix}^T, I_2 = \begin{bmatrix} 0 & 0 & I & -I & 0 \end{bmatrix}^T.
\end{aligned}$$

One can write (5.39) as:

$$\upsilon + (\delta\phi_1 + I_1 R_2)\, R_2^{-1}(\delta\phi_1 + I_1 R_2)^T - I_1 R_2 I_1^T + (\delta\phi_2 + I_2 R_2)\, R_2^{-1}(\delta\phi_2 + I_2 R_2)^T - I_2 R_2 I_2^T < 0. \tag{5.40}$$

Further, following Lemma 1.2, substituting the free variables as $M_i = {M_i}^T = -N_i = -{N_i}^T = -\delta^{-1}R_2$, the above stability condition yields.

$$\bar{\upsilon} < 0, \tag{5.41}$$

where

$$\bar{\upsilon} = [\bar{\upsilon}_{ij}]_{i,j=1,\dots,5}, \bar{\upsilon}_{11} = \upsilon_{11}, \bar{\upsilon}_{12} = \upsilon_{12}, \bar{\upsilon}_{13} = \upsilon_{13}, \bar{\upsilon}_{14} = \upsilon_{14}, \bar{\upsilon}_{15} = \upsilon_{15},$$

$$\begin{aligned}
\bar{\upsilon}_{22} &= -(Q_2 - Q_4) - R_1 - R_2 + \sum_{k=1}^{3} \varepsilon_k S_2 D_k D_k^T S_2^T, \\
\bar{\upsilon}_{23} &= S_2 B_2 K + R_2 + \sum_{k=1}^{3} \varepsilon_k S_2 D_k D_k^T S_3^T, \bar{\upsilon}_{24} = \upsilon_{24}, \bar{\upsilon}_{25} = \upsilon_{25}, \\
\bar{\upsilon}_{33} &= -\sum_{k=3}^{4} Q_k + S_3 B_2 K + K^T B_2^T S_3^T + \varepsilon_2^{-1} K^T E_2^T E_2 K - 2R_2 + \sum_{k=1}^{3} \varepsilon_k S_3 D_k D_k^T S_3^T, \\
\bar{\upsilon}_{34} &= K^T B_2^T S_4^T + R_2 + \sum_{k=1}^{3} \varepsilon_k S_3 D_k D_k^T S_4^T, \bar{\upsilon}_{35} = \upsilon_{35}, \\
\bar{\upsilon}_{44} &= -Q_1 - R_2 + \sum_{k=1}^{3} \varepsilon_k S_4 D_k D_k^T S_4^T, \bar{\upsilon}_{45} = \upsilon_{45}, \bar{\upsilon}_{55} = \upsilon_{55}.
\end{aligned}$$

For linearization, considering S_2, S_3, S_4 and S_5 as: $S_2 = \lambda S_1$, $S_3 = \beta S_1$, $S_4 = \gamma S_1$, $S_5 = \alpha S_1$ and then, pre- and post-multiplying by $diag\left\{S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \quad S_1^{-1}\right\}$ and its transpose respectively, and subsequently adopting the change of variables

$$\begin{aligned}
\bar{S}_1 &= S_1^{-1}, \bar{P} = \bar{S}_1 P \bar{S}_1^T, \bar{M}_i = \bar{S}_1 M_i \bar{S}_1^T, \bar{N}_i = \bar{S}_1 N_i \bar{S}_1^T, i = 1, 2, \bar{Q}_j = \bar{S}_1 Q_j \bar{S}_1^T, j = 1, ..4, \\
\bar{Y} &= K \bar{S}_1^T.
\end{aligned}$$

One obtains (5.36). □

5.4.2 Numerical examples

In this section some numerical examples are presented to validate proposed theorem in the previous section.

Example 5.3. Consider the Example 5.1, the maximum η (η_{max}) is achieved using Theorem 5.2 to be 10.0480. For this system, the tuning parameters λ, β, γ and α are tuned at 0.0434, −0.1685, 0.1236 and 0.2114 respectively by a controller $K = \begin{bmatrix} -13.9843 & -3.6384 \end{bmatrix}$. The obtained η_{max} using Theorem 5.2 is compared with that obtained using Theorem 5.1 in Table 5.3. To validate the proposed Theorem 5.2, the simulation results using the designed controller ($K = \begin{bmatrix} -13.9843 & -3.6384 \end{bmatrix}$) with initial condition $x(t) = [3, -2]$, $t \in [-10.0480, 0]$ is presented in Fig. 5.3. This results shows that the states of the closed-loop system are stable at $\eta_{max} = 10.0480$.

Table 5.3: Comparison of robustness η_{max}

Approach	η_{max}	Structure of $u(t)$
Theorem 5.1	10.0048	$u(t) = Kx(t)$
Theorem 5.2	10.0480	$u(t) = Kx(t)$

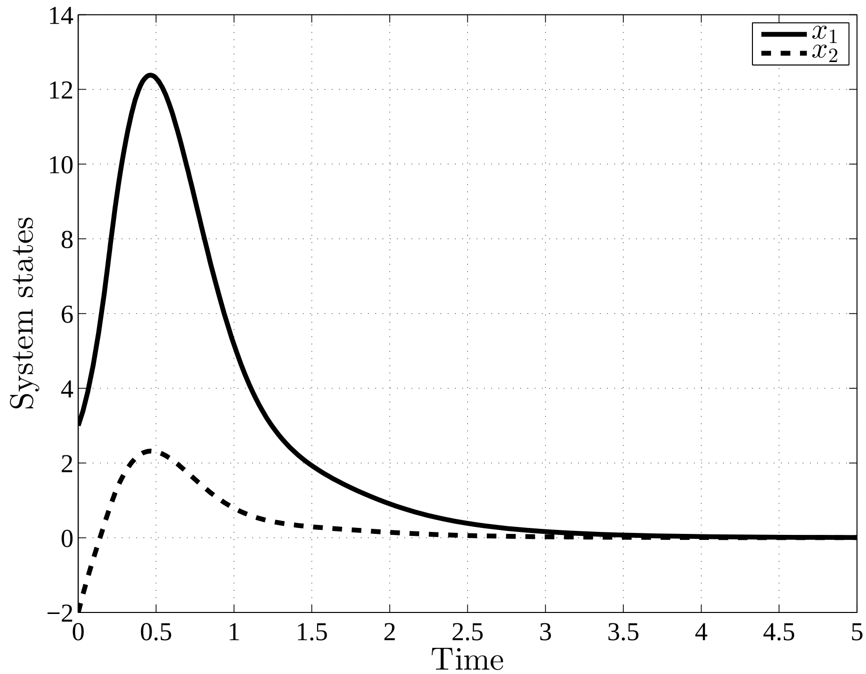

Figure 5.3: Variation of system states with respect to time for Example 5.3

Example 5.4. Next, consider system (5.1) with [108]

$$
\begin{aligned}
A_0 &= \begin{bmatrix} -0.08 & -0.03 & 0.2 \\ 0.2 & -0.04 & -0.005 \\ -0.06 & 0.2 & -0.07 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} -0.1 \\ -0.2 \\ 0.1 \end{bmatrix}, \\
D_0 &= D_1 = D_2 = 0 \quad and \quad E_0 = E_1 = E_2 = 0.
\end{aligned}
$$

For this system, using Theorem 5.2, the maximum delay margin $(\bar{h})$ is obtained to be 9.2541. The corresponding tuning parameters λ, β, γ and α are tuned at -1.0092, 0.0017, 0.0075 and 89.6665 respectively that yields a controller $K = [0.4419 \quad 0.5597 \quad 0.4806]$. A comparison of the present results with existing one in [108] is presented in Table 5.4 that shows the less conservativeness of the developed criterion.

Table 5.4: Maximum Tolerable Delay Bound ($\bar{h}$)

Approach	$\bar{h}$
[108]	5
Theorem 5.1	9.1626
Theorem 5.2	9.2541

From the above two examples, it may be noted that the static state-feedback stabilization criterion using delay decomposition approach obtained in the previous section is slightly less conservative as compared static state-feedback stabilization criterion derived without delay-decomposition approach proposed in Chapter 2. It is expected that the former one will reduce the conservativeness extensively. However, during the process of designing the controller gain matrix (K), a nonlinear matrix inequality of the form (5.44) is obtained. To linearize such nonlinear matrix inequality, the matrix variables are approximated as $S_2 = \lambda S_1$, $S_3 = \beta S_1$, $S_4 = \gamma S_1$, $S_5 = \alpha S_1$. The involvement of this approximation process might be a reason that considerable reduction in conservativeness could not be attained. The further reduction of conservatism can be investigated by using a controller of higher degree of freedom.

5.5 Stabilization criterion using PI-type controller

Next, we consider designing PI-type state feedback controller for systems with input delays

5.5.1 The PI controller

A stabilization criterion is derived in this section for system (5.1) using a state-feedback PI-controller of the form

$$u(t) = K_p x(t) + K_I \int_0^t x(\theta) d\theta, \tag{5.42}$$

where K_p and K_I are the control gains to be designed so that the system is stabilized. Consider

$$\dot{z}(t) = x(t). \tag{5.43}$$

and

$$\bar{x}(t) = \begin{bmatrix} x^T(t) & z^T(t) \end{bmatrix}^T. \tag{5.44}$$

So, the control input (5.42) can now be written as:

$$u(t) = \begin{bmatrix} K_p & K_I \end{bmatrix} \bar{x}(t). \tag{5.45}$$

Using the controller (5.45) in (5.1), the augmented closed-loop system can be written as:

$$\dot{\bar{x}}(t) = \bar{A}_0\bar{x}(t) + \bar{B}_1 K\bar{x}(t) + \bar{B}_2 K\bar{x}(t-h), \tag{5.46}$$

where

$$\begin{aligned} \bar{A}_0 &= \begin{bmatrix} (A_0 + \Delta A_0(t)) & 0 \\ I & 0 \end{bmatrix}, \bar{B}_0 = \begin{bmatrix} (B_1 + \Delta B_1(t)) \\ 0 \end{bmatrix}, \bar{B}_1 = \begin{bmatrix} (B_2 + \Delta B_2(t)) \\ 0 \end{bmatrix}, \\ K &= \begin{bmatrix} K_p & K_I \end{bmatrix}. \end{aligned}$$

The following lemma is used to derive main stability criterion.

Lemma 5.2. *For any arbitrary matrices S_1, S_2, S_3, S_4 and S_5 the following condition holds:*

$$\begin{gathered} 2\xi^T(t)\begin{bmatrix} S_1^T & S_2^T & S_3^T & S_4^T & S_5^T \end{bmatrix}^T \\ \left\{-\dot{\bar{x}}(t) + \bar{A}_0\bar{x}(t) + \bar{B}_1 K\bar{x}(t) + \bar{B}_2 K\bar{x}(t-h)\right\} = 0, \end{gathered} \tag{5.47}$$

where

$$\xi(t) = \begin{bmatrix} \bar{x}^T(t) & \bar{x}^T(t-h_1) & \bar{x}^T(t-h(t)) & \bar{x}^T(t-h_2) & \dot{\bar{x}}^T(t) \end{bmatrix}^T.$$

Proof: Using (5.46), one can write

$$\left\{-\dot{\bar{x}}(t) + \bar{A}_0\bar{x}(t) + \bar{B}_1 K\bar{x}(t) + \bar{B}_2 K\bar{x}(t-h)\right\} = 0. \tag{5.48}$$

One obtains (5.47) by multiplying $2\xi^T(t)\begin{bmatrix} S_1^T & S_2^T & S_3^T & S_4^T & S_5^T \end{bmatrix}^T$ with (5.48).

The objective of this section is to design a controller of the form (5.42) for the system (5.1).

5.5.2 Stabilization criterion

The following stabilization criterion is used to design the PI-controller for (5.1).

Theorem 5.3. *System (5.46) is stable if, for arbitrarily chosen λ, β, γ and α, there exist matrices $\bar{P} > 0$, $\bar{Q}_j > 0$, $j = 1, \ldots, 4$, $\bar{R}_i > 0$, and arbitrary matrices $\bar{S}_1$, $\bar{M}_i$, $\bar{N}_i$, $i = 1, 2$, that satisfy the following LMI:*

$$\begin{bmatrix} \bar{\Theta} & \bar{\phi}_l \\ * & -\bar{R}_2 \end{bmatrix} < 0, \quad l = 1, 2, \tag{5.49}$$

where

$$\begin{aligned}
\bar{\phi}_1 &= \begin{bmatrix} 0 & \bar{M}_1^T & \bar{N}_1^T & 0 & 0 \end{bmatrix}^T, \bar{\phi}_2 = \begin{bmatrix} 0 & 0 & \bar{M}_2^T & \bar{N}_2^T & 0 \end{bmatrix}^T, \\
\bar{\Theta} &= [\bar{\Theta}_{ij}]_{i,j=1,..,5} \quad with \quad \bar{\Theta}_{11} = \sum_{i=1}^{3} \bar{Q}_i - \bar{R}_1 + \bar{A}_0 \bar{S}_1^T + \bar{S}_1 \bar{A}_0^T + \bar{B}_1 \bar{Y} + \bar{Y}^T \bar{B}_1^T, \\
\Theta_{12} &= \bar{R}_1 + \lambda \bar{S}_1 \bar{A}_0^T + \lambda \bar{Y}^T \bar{B}_1^T, \Theta_{13} = \bar{B}_2 \bar{Y} + \beta \bar{S}_1 \bar{A}_0^T + \beta \bar{Y}^T \bar{B}_1^T, \\
\Theta_{14} &= \gamma \bar{S}_1 \bar{A}_0^T + \gamma \bar{Y}^T \bar{B}_1^T, \Theta_{15} = \bar{P} - \bar{S}_1^T + \alpha \bar{S}_1 \bar{A}_0^T + \alpha \bar{Y}^T \bar{B}_1^T, \\
\Theta_{22} &= -\left(\bar{Q}_1 - \bar{Q}_4\right) - \bar{R}_1 + \bar{h}^{-1}\left[\bar{M}_1 + \bar{M}_1^T\right], \Theta_{23} = \lambda \bar{B}_2 \bar{Y} + \bar{h}^{-1}\left[-\bar{M}_1 + \bar{N}_1^T\right], \Theta_{24} = 0, \\
\Theta_{25} &= -\lambda \bar{S}_1^T, \Theta_{33} = -\sum_{i=3}^{4} \bar{Q}_i + \beta \bar{B}_2 \bar{Y} + \beta \bar{Y}^T \bar{B}_2^T + \bar{h}^{-1}\left[-\bar{N}_1 - \bar{N}_1^T\right] + \bar{h}^{-1}\left[\bar{M}_2 + \bar{M}_2^T\right], \\
\Theta_{34} &= \gamma \bar{Y}^T \bar{B}_2^T + \bar{h}^{-1}\left[-\bar{M}_2 + \bar{N}_2^T\right], \Theta_{35} = -\beta \bar{S}_1^T + \alpha \bar{Y}^T \bar{B}_2^T, \\
\Theta_{44} &= -\bar{Q}_2 + \bar{h}^{-1}\left[-\bar{N}_2 - \bar{N}_2^T\right], \Theta_{45} = -\gamma \bar{S}_1^T, \Theta_{55} = h_1^2 \bar{R}_1 + \bar{R}_2 - \alpha \bar{S}_1^T - \alpha \bar{S}_1, \\
\bar{S}_1 &= S_1^{-1}, \bar{P} = \bar{S}_1 P \bar{S}_1^T, \bar{Q}_i = \bar{S}_1 Q_i \bar{S}_1^T, i = 1, ..., 4, \bar{M}_j = \bar{S}_1 M_j \bar{S}_1^T, \bar{S}_1 N_j \bar{S}_1^T = \bar{N}_j, j = 1, 2, \\
\bar{Y} &= K \bar{S}_1^T.
\end{aligned}$$

Proof: Consider the following Lyapunov-Krasovskii functional

$$\begin{aligned}
V(t) = \bar{x}^T(t) P \bar{x}(t) + \sum_{i=1}^{2} \int_{t-h_i}^{t} \bar{x}^T(\theta) Q_i \bar{x}(\theta) d\theta + \int_{t-h}^{t} \bar{x}^T(\theta) Q_3 \bar{x}(\theta) d\theta \\
+ \int_{t-h}^{t-h_1} \bar{x}^T(\theta) Q_4 \bar{x}(\theta) d\theta + h_1 \int_{t-h_1}^{t} \int_{\theta}^{t} \dot{\bar{x}}^T(\varphi) R_1 \dot{\bar{x}}(\varphi) d\varphi d\theta + \bar{h}^{-1} \int_{t-h_2}^{t-h_1} \int_{\theta}^{t} \dot{\bar{x}}^T(\varphi) R_2 \dot{\bar{x}}(\varphi) d\varphi d\theta.
\end{aligned} \tag{5.50}$$

Differentiating (5.50) with respect to time along the state trajectory of (5.48) is

$$\begin{aligned}\dot{V}(t) = 2\bar{x}^T(t)P\dot{\bar{x}}(t) + \sum_{i=1}^{3}\bar{x}^T(t)Q_i\bar{x}(t) - \sum_{i=3}^{4}(1-\mu)\bar{x}^T(t-h(t))Q_i\bar{x}(t-h(t))\\ -\bar{x}^T(t-h_1)(Q_1-Q_4)\bar{x}(t-h_1) - \bar{x}^T(t-h_2)Q_2\bar{x}(t-h_2)\\ +\dot{\bar{x}}^T(t)(h_1^2R_1+R_2)\dot{\bar{x}}(t) - h_1\int_{t-h_1}^{t}\dot{\bar{x}}^T(\theta)R_1\dot{\bar{x}}(\theta)d\theta - \bar{h}^{-1}\int_{t-h_2}^{t-h_1}\dot{\bar{x}}^T(\theta)R_2\dot{\bar{x}}(\theta)d\theta.\end{aligned} \tag{5.51}$$

The stability of the (5.48) can be analyzed by checking $\dot{V}(t)$ is less than zero or not, the R.H.S. of (5.47) is added to (5.51). Then, it becomes

$$\begin{aligned}2\xi^T(t)\begin{bmatrix}S_1^T & S_2^T & S_3^T & S_4^T & S_5^T\end{bmatrix}^T\left\{-\dot{\bar{x}}(t)+\bar{A}_0\bar{x}(t)+\bar{B}_1K\bar{x}(t)+\bar{B}_2K\bar{x}(t-h)\right\}\\ +2\bar{x}^T(t)P\dot{\bar{x}}(t) + \sum_{i=1}^{3}\bar{x}^T(t)Q_i\bar{x}(t) - \sum_{i=3}^{4}\bar{x}^T(t-h)Q_i\bar{x}(t-h)\\ -\bar{x}^T(t-h_1)(Q_1-Q_4)\bar{x}(t-h_1) - \bar{x}^T(t-h_2)Q_2\bar{x}(t-h_2)\\ +\dot{\bar{x}}^T(t)(h_1^2R_1+R_2)\dot{\bar{x}}(t) - h_1\int_{t-h_1}^{t}\dot{\bar{x}}^T(\theta)R_1\dot{\bar{x}}(\theta)d\theta - \bar{h}^{-1}\int_{t-h_2}^{t-h_1}\dot{\bar{x}}^T(\theta)R_2\dot{\bar{x}}(\theta)d\theta.\end{aligned} \tag{5.52}$$

Approximating the two integral terms in the RHS of (5.52) using Lemma 1.2, (5.52) can be written as

$$\xi^T(t)\left\{\Theta + \rho\phi_1R_2^{-1}\phi_1^T + (1-\rho)\phi_2R_2^{-1}\phi_2^T\right\}\xi(t), \tag{5.53}$$

Note that, (5.53) is polytope of matrices and is negative definite if it's two certain vertices are negative definite individually. Then, the stability requirement can be written as:

$$\Theta + \phi_lR_2^{-1}\phi_l^T < 0, \quad l = 1, 2. \tag{5.54}$$

Finally, using Schur Complement on (5.54), one obtains

$$\begin{bmatrix}\Theta & \phi_l\\ * & -R_2\end{bmatrix} < 0, \quad l = 1, 2, \tag{5.55}$$

where

$$\phi_1 = \begin{bmatrix}0 & M_1^T & N_1^T & 0 & 0\end{bmatrix}^T, \phi_2 = \begin{bmatrix}0 & 0 & M_2^T & N_2^T & 0\end{bmatrix}^T,$$

$$
\begin{aligned}
\Theta &= [\Theta_{ij}]_{i,j=1,..,5} \quad with \\
\Theta_{11} &= \sum_{i=1}^{3} Q_i - R_1 + S_1\bar{A}_0 + \bar{A}_0^T S_1^T + S_1\bar{B}_1 K + K^T\bar{B}_1^T S_1^T, \\
\Theta_{12} &= R_1 + \bar{A}_0^T S_2^T + K^T\bar{B}_1^T S_2^T, \Theta_{13} = S_1\bar{B}_2 K + \bar{A}_0^T S_3^T + K^T\bar{B}_1^T S_3^T, \\
\Theta_{14} &= \bar{A}_0^T S_4^T + K^T\bar{B}_1^T S_4^T, \Theta_{15} = P - S_1 + \bar{A}_0^T S_5^T + K^T\bar{B}_1^T S_5^T, \\
\Theta_{22} &= -(Q_1 - Q_4) - R_1 + \bar{h}^{-1}\left[M_1 + M_1^T\right], \\
\Theta_{23} &= S_2\bar{B}_2 K + \bar{h}^{-1}\left[-M_1 + N_1^T\right], \Theta_{24} = 0, \Theta_{25} = -S_2, \\
\Theta_{33} &= -\sum_{i=3}^{4} Q_i + S_3\bar{B}_2 K + K^T\bar{B}_2^T S_3^T + \bar{h}^{-1}\left[-N_1 - N_1^T\right] + \bar{h}^{-1}\left[M_2 + M_2^T\right], \\
\Theta_{34} &= K^T\bar{B}_2^T S_4^T + \bar{h}^{-1}\left[-M_2 + N_2^T\right], \Theta_{35} = -S_3 + K^T\bar{B}_2^T S_5^T, \\
\Theta_{44} &= -Q_2 + \bar{h}^{-1}\left[-N_2 - N_2^T\right], \Theta_{45} = -S_4, \Theta_{55} = h_1^2 R_1 + R_2 - S_5 - S_5^T.
\end{aligned}
$$

The nonlinear terms in (5.55) can be eliminated by considering S_2, S_3, S_4 and S_5 as:

$$S_2 = \lambda S_1, S_3 = \beta S_1, S_4 = \gamma S_1, S_5 = \alpha S_1.$$

and then pre- and post-multiplying (5.55) by

$$diag\left\{S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \quad S_1^{-1} \quad S_1^{-1}\right\}$$

and its transpose respectively, and subsequently adopting the change of variables

$$
\begin{aligned}
\bar{S}_1 &= S_1^{-1}, \bar{P} = \bar{S}_1 P\bar{S}_1^T, \bar{M}_i = \bar{S}_1 M_i\bar{S}_1^T, \bar{N}_i = \bar{S}_1 N_i\bar{S}_1^T, i = 1,2, \bar{Q}_j = \bar{S}_1 Q_j\bar{S}_1^T, \\
j &= 1,\ldots,4, \bar{Y} = K\bar{S}_1^T.
\end{aligned}
$$

With the above procedure, one finally obtains (5.49).

To verify the above criterion proposed in this section, two numerical examples are considered in the next section.

5.5.3 Numerical examples

Some numerical examples are presented in this section to validate the stabilization criterion developed in the previous section.

Example 5.5. Consider the system in Example 5.1. Also recollect that the maximum η (η_{max}) is achieved using Theorem 5.3 to be 28.7690. For this same system, using LMI

(5.49) the tuning parameters λ, β, γ and α are tuned as 2.5107, −4.4885, 2.9221 and 0.9954 respectively with a controller $K = \begin{bmatrix} K_p & K_I \end{bmatrix}$, where $K_p = \begin{bmatrix} -33.1133 & -4.7441 \end{bmatrix}$ and $K_I = \begin{bmatrix} -0.0397 & -0.0008 \end{bmatrix}$, which is more robust than the existing controllers in [14, 174] since η_{max} obtained for this case is quite large. A comparison of all the results is presented in Table 5.3. The simulation result (norm of the states of the system) is presented with $x(t) = [5, -2, -3, -1]$, $t \in [-28.7690, 0]$ initial condition in Fig. 5.4 using the PI-type controller for $\eta_{max} = 28.7690$. The results shows that the states are stable.

Table 5.5: Comparison of robustness η_{max}

Approach	η_{max}	Structure of $u(t)$
Theorem 5.1	10.0048	$u(t) = Kx(t)$
Theorem 5.2	10.0480	$u(t) = Kx(t)$
Theorem 5.3	28.7690	$u(t) = K_p x(t) + K_I \int_0^t x(\theta)d\theta$ where $K_p = \begin{bmatrix} -33.1133 & -4.7441 \end{bmatrix}$ and $K_I = \begin{bmatrix} -0.0397 & -0.0008 \end{bmatrix}$

Example 5.6. Consider another system of (5.1) with [14]

$$\begin{aligned} \dot{x}(t) &= (A_0 + \Delta A_0)x(t) + B_1 u(t) + B_2 u(t-h), \quad t \geq 0, \\ x(0) &= x_0, \quad u(t) = \phi(t), \quad t \in [-0.4, 0], \end{aligned} \tag{5.56}$$

where

$$A_0 = \begin{bmatrix} 0 & 0 \\ 1 & -5 \end{bmatrix}, \quad \Delta A_0 = \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |q| \leq \eta.$$

In this case, the η_{max} is obtained using Theorem 5.3 to be 1.8524. The tuning parameters λ, β, γ and α are tuned at 1.2887, 0.1741, 0.3952 and 2.9147 respectively by a controller $K = \begin{bmatrix} K_p & K_I \end{bmatrix}$, where $K_p = \begin{bmatrix} -2.3661 & -0.0035 \end{bmatrix}$ and $K_I = \begin{bmatrix} -0.0247 & -0.0074 \end{bmatrix}$, which is also more robust than the existing controllers in [14, 174]. A comparison with the existing results is presented in Table 5.6. From this analysis, one can easily conclude that the result obtained using Theorem 5.3 is less conservative than that of Theorem 5.1 and Theorem 5.2.

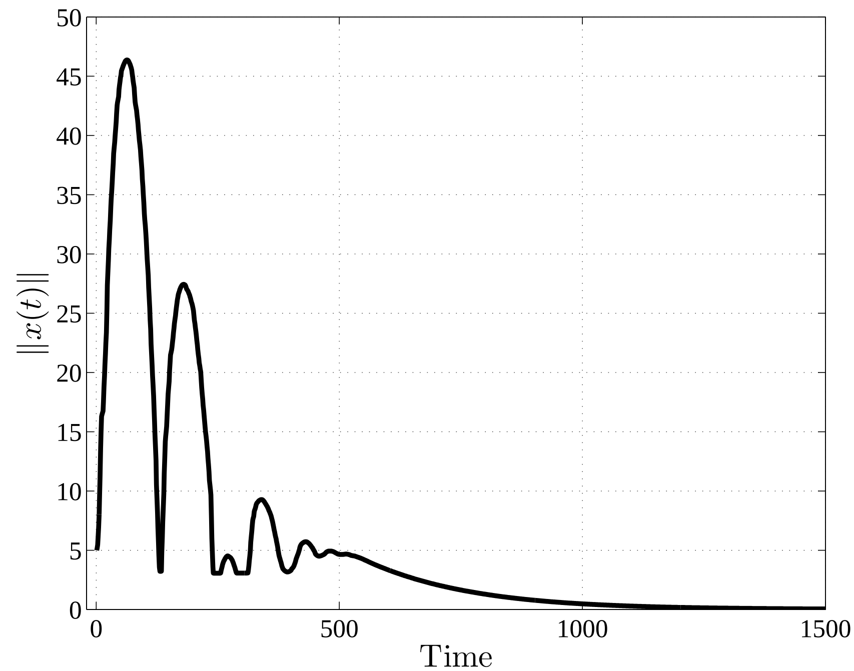

Figure 5.4: Variation of norm of the state vector with respect to time for Example 5.5

Table 5.6: Comparison of robustness η_{max}

Approach	η_{max}
[174]	0.5998
[14]	1.4120
Theorem 5.1	1.2175
Theorem 5.2	1.2204
Theorem 5.3	1.8524

5.6 Summary

In this chapter:

- An improved robust delay-dependent stabilizing criterion has been obtained by using a simple static state feedback controller. The obtained result is less conservative than that of [174] but conservative to [14]. Though the result is slightly conservative to [14], the present controller is comparatively simple in structure and hence easier to implement.

- A simple linearization technique is adopted for linearizing the non-linear terms in the stability criterion to obtain the LMI conditions.

- To improve the robustness, the delay-decomposition approach proposed in Chapter 2 is implemented. But the decomposition technique is not able to reduce the conservativeness due to the linearization technique.

- Finally, This chapter includes an improved robust delay-dependent stabilizing criterion by using a PI-type controller. By adding the integral control action with simple memoryless controller, the degree of the freedom is increased as a result of which the robustness is increased. The obtained new robust stabilizing criterion is less conservative than that of the existing results [14]. Numerical examples are considered to show the effectiveness of the criterion than that of criteria derived using a simple static state feedback control.

CHAPTER 6

Stabilization using dynamic state-feedback controller with artificial delays

This chapter presents investigations on a dynamic state-feedback controller with artificial delays that improves tolerable delay margin for systems with input-output delays. Using an iterative pole placement technique for time-delay systems, the effect of introducing artificial delay in the controller dynamics is studied. It is observed that such a controller improves the tolerable delay margins compared to its static or even simple dynamic counterpart.

6.1 Introduction

Time-delay is inherent to many feedback control systems owing to the fact that information takes finite time to get transported. Often, delays appear in the feedback loop due to the time taken in (i) measuring outputs (ii) computing control actions and (iii) actuating the plant. Such delays in the feedback loop are, in general, destabilizing [43]. However, it is also possible that purposeful use of artificial delays in the controller may improve stability of certain systems, e.g., (i) use of an appropriate adjustment of the spindle speed helps in tuning the delay to avoid chattering in metal machining in a milling process [152], (ii) use of delay may yield better purchasing and stocking decisions in supply chain management [59]. Such stabilizing effect of delays is a motivation to many researchers to exploit the possibilities of using them with benefits.

This chapter considers the problem of stabilizing systems with Input and Output (IO) delays as shown in Fig. 6.1. Time taken in measuring the output signal and thereby receiving at the controller is called as the output delay (h_s), whereas the sending time for the control signal from the controller to the actuator is the input delay (h_a). For such systems, if one uses a state-feedback controller then the delay in the feedback loop may be represented as $h_{total} = h_a + h_s$ [86].

For an illustration, consider a scalar system of the form

$$\dot{x}(t) = ax(t) + u(t - h_a), \tag{6.1}$$

It is well known that using a static state-feedback controller of the form $u(t) = k_s x(t - h_s)$, where k_s is the control gain, system (6.1) can be stabilized till $a\,(h_a + h_s) < 1$ [108]. However, if one uses an observer based controller of the form

$$\dot{\hat{x}}(t) = a\hat{x}(t) + k\hat{x}(t - h_a) + l\hat{x}(t - h_s) - lx(t - h_s), \tag{6.2}$$

where $\hat{x}(t)$ is the estimate of the state, l is the observer gain, then the scalar system (6.1) can be stabilized till $ah_a < 1$ and $ah_s < 1$ [108], which is an improvement over the static feedback one. However, implementing such a controller is difficult since one has to obtain accurate information of the two delays, which is impractical specifically when these delays are uncertain or time-varying.

From this perspective, it may be intuited that dynamic controller with delay might have stability improvement ability for time-delay systems. Note that the inclusion of delay in such controllers is important in addition to the dynamicness. Since, similar to systems without time delays, simple dynamic controllers without time delay doesn't have any stability improvement ability as compared to static controllers typically for state-feedback case [108, 142].

From the above discussion, it may be perceived that dynamics and state delays combinedly in controllers may help in improving the tolerable delay bound. Question that now arises is whether the controller dynamics, its state delays or both of them contribute to this improvement. This chapter attempts to address this question and proposes a dynamic state-feedback controller with state delays that improves tolerable delay bound in the feedback loop further. It is to mention here that this work does not investigate the stabilizing ability of controller with time delays for systems that are not otherwise stabilizable, as it has been attempted in [143]. Rather, it looks into the possibility of tolerable delay margin improvement for systems that are conventionally stabilizable. To investigate the stabilizing ability of the delayed state-feedback dynamic controller, three variants of dynamic controller structure are studied for linear time-invariant plant. However, the design of controller parameters is a challenging issue for time-delay systems. A well-known Continuous Pole Placement Technique (CPPT) [108] for time-delay system is used for designing the controller. A brief description of this technique is presented in subsequent section of the chapter. To implement this algorithm, DDE-BIFTOOL [23] is used.

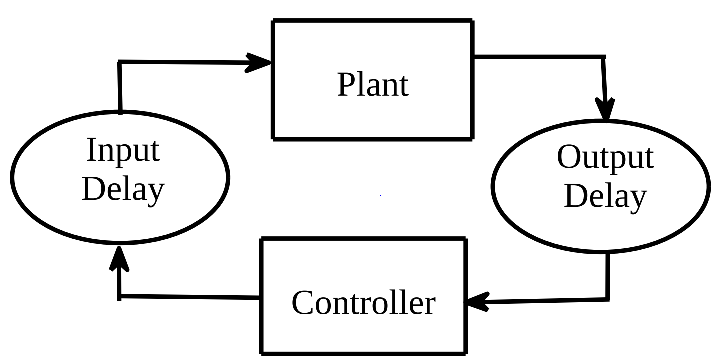

Figure 6.1: Feedback control system with input-output delays

6.2 Problem consideration

The system considered in this chapter is shown in Fig. 6.1. The plant dynamics with input delay is represented as:

$$\dot{x}_p(t) = A_p x_p(t) + B_p u_p(t - h_a), \tag{6.3}$$

where $x_p(t) \in \Re^{n_p}$ is the state, $u_p(t) \in \Re^{m_p}$ is the control input; A_p and B_p are constant matrices of appropriate dimensions; h_a is the input delay of the systems. For stabilizing such systems, we consider the following controller types with $h_s(\geq 0)$ delay in the output:
Type I: Simple dynamic controller

$$\dot{x}_c(t) = A_{c0} x_c(t) + C_c x_p(t - h_s), \quad u_p(t) = x_c(t); \tag{6.4}$$

Type II: Dynamic controller with a state-delay

$$\dot{x}_c(t) = A_{c0} x_c(t) + A_{c1} x_c(t - h_1) + C_c x_p(t - h_s), \quad u_p(t) = x_c(t); \tag{6.5}$$

Type III: Dynamic controller with two state-delays

$$\begin{aligned} \dot{x}_c(t) &= A_{c0} x_c(t) + A_{c1} x_c(t - h_1) + A_{c2} x_c(t - h_2) + C_c x_p(t - h_s), \\ u_p(t) &= x_c(t); \end{aligned} \tag{6.6}$$

where $x_c(t) \in \Re^{m_p}$ is the state of the dynamic controller and A_{c0}, A_{c1}, A_{c2} and C_c are the controller matrices to be designed.
Stabilization using Type I is of interest to study the effect of controller dynamics on improvement in tolerable delay ranges whereas the same for Type II corresponds to the effect of both the dynamics and controller state-delay. Comparison of the stabilizing ability of Type III explores whether use of more than one delay in the controller states has any further effect. It may be noted that the controller of Type III is similar to the observer based controller. However, the delays h_1 and h_2 may take different values other than the IO delays and may be chosen appropriately.

The closed-loop system for the Type-I controller, (6.3) along with (6.4), may be written as:

$$\dot{\xi}(t) = A\xi(t) + D\xi(t - h_a) + E\xi(t - h_s), \tag{6.7}$$

The closed-loop system for the Type-II controller, (6.3) along with (6.5), may be written as:

$$\dot{\xi}(t) = A\xi(t) + B\xi(t-h_1) + D\xi(t-h_a) + E\xi(t-h_s), \tag{6.8}$$

The closed-loop system for the Type-III controller, (6.3) along with (6.6), may be written as:

$$\dot{\xi}(t) = A\xi(t) + B\xi(t-h_1) + C\xi(t-h_2) + D\xi(t-h_a) + E\xi(t-h_s), \tag{6.9}$$

where

$$\xi(t) = \begin{bmatrix} x_p(t) \\ x_c(t) \end{bmatrix}, A = \begin{bmatrix} A_p & 0 \\ 0 & A_{c0} \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & A_{c1} \end{bmatrix},$$
$$C = \begin{bmatrix} 0 & 0 \\ 0 & A_{c2} \end{bmatrix}, D = \begin{bmatrix} 0 & B_p \\ 0 & 0 \end{bmatrix}, E = \begin{bmatrix} 0 & 0 \\ C_c & 0 \end{bmatrix}.$$

Note that, the complexity of the closed loop system dynamics depends on the introduction of number artificial delays in the controller dynamics.

6.3 Continuous Pole-Placement Technique (CPPT)

For being infinite dimensional system, stability of time-delay systems are often addressed using computational approaches, e.g., using frequency sweeping test [43], stability charts [7]. Stabilization, i.e. controller design guaranteeing stability is expectedly more complex. Lyapunov approaches lead to computationally efficient criteria in terms of Riccati equations [148] or Linear Matrix Inequalities [33, 38], but they are indeed conservative so far. The finite spectrum assignment technique using a predictor based controller [79] does not meet the present requirement. To this end, a numerical stabilization technique is available, called as Continuous Pole Placement Technique (CPPT) using which stabilizing control gains may be obtained iteratively [108]. Since, there exists no closed form solution for stabilization of time-delay systems, we use this numerical procedure just to find the existence of, possibly superior, controllers. We next present CPPT briefly. Details on convergence and performance of the algorithm have been discussed in [108].

Consider C_r^+ represents the closed right half plane of a vertical line through $r \in R$ and contains poles with $R(\lambda) \geq r$. Further, C_r^- is complementary to C_r^+. Note that, the number of poles for a time-delay system on C_r^+ is finite and computationally tractable [7]. Software packages are available that can be used readily for computing them, e.g., DDE-

BIFTOOL [23]. This tool has been used to compute the rate of synchronization in biological networks with time-delays [165], to design higher truncated predictor for linear systems [180] and to study flute-like instruments which are modelled as a delay dynamical system [162].

let us define $K \in \Re^q$ consisting of the q number of controller parameters. The dimension of K depends on the consideration on the order of the dynamic controller. While, for a chosen dynamic controller, the dimension of K is fixed, its parameters may be chosen differently depending on one's choice. However corresponding to a particular choice, the computation of sensitivity matrix changes. For designing K, the designer is required to arrange all the controller gain parameters in a row matrix. To have a clear view on this, some example cases are demonstrated as follows:

1. For Type I, the simple dynamic controller (6.4), of the structure $\dot{x}_c(t) = a_{c0}x_c(t) + c_c x_p(t - h_s), u_p(t) = x_c(t)$, the controller gains parameters are a_{c0} and c_c. For this, a choice of K is $K = [a_{c0}, c_c]$.

2. Similarly a Type III dynamic controller, of the structure $\dot{x}_c(t) = a_{c0}x_c(t) + a_{c1}x_c(t - h_1) + a_{c2}x_c(t - h_2) + c_c x_p(t - h_s), u_p(t) = x_c(t)$ the designed parameters are a_{c0}, a_{c1}, a_{c2} and c_c. For this, a choice of K is $K = [a_{c0}, a_{c1}, a_{c2}, c_c]$.

Now, for stabilization, one requires to obtain controller parameters K for which all closed-loop poles are placed in C_0^-. A sensitivity matrix $S = [S_{i,j}] \in \Re^{q \times m}$, where $S_{i,j} = \frac{\partial \lambda_i}{\partial k_j}$, λ_i being the i^{th} rightmost pole and k_j the j^{th} element of K. Then the CPPT with a slight modification to the one presented in [108] for obtaining a stabilizing controller is presented in the following.

Algorithm 6.1.

Step 1: *Initialize controller parameters K. Set a $\bar{r}$ for which the poles in $C_{\bar{r}}^+$ will be regulated. Let the number of poles in $C_{\bar{r}}^+$ be m and set a desired change in the regulated poles as $\delta_\lambda \in \Re^m$.*

Step 2: *Update m, δ_λ, and compute S.*

Step 3: *Compute desired change in controller parameters as $\delta_K = S^\dagger \delta_\lambda$, where $S^\dagger$ is the Moore-Penrose inverse of S, and update K as $K = K + \delta_K$.*

Step 4: *Check whether the rightmost pole is placed to the left-half plane. If not then go to Step 2 until a certain number of iterations, else exit declaring stabilized or not.*

Remark 6.1. *Note that, m must be less than or at most equal to q. Therefore, a combination of $\bar{r}$ and initial K may be chosen so that the number of poles in $C_{\bar{r}}^+$ is $\leq q$. Moreover,* *since mere stabilization is concerned in this chapter, one may consider only the real parts of the poles for regulation, i.e., for a complex pole pair the number of controlled poles (real component) would be one.*

To this end, one need to compute $\frac{\partial \lambda_i}{\partial k_j}$ to obtain S. For the purpose, let i^{th} root of the characteristic equation of (6.9) be λ_i and hence it satisfies

$$\left\{ \lambda_i I - A - Be^{-\lambda_i h_1} - Ce^{-\lambda_i h_2} - De^{-\lambda_i h_a} - Ee^{-\lambda_i h_s} \right\} \times v_i = 0, \tag{6.10}$$

where v_i is the corresponding eigenvector of (6.9).
Using a normalizing function for v_i, one may write

$$f_{\aleph}(v_i) = 0. \tag{6.11}$$

Taking derivative of (6.10) and (6.11) with respect to k_j, one obtains

$$\begin{bmatrix} \psi_1 & \psi_2 \\ \frac{df_{\aleph}^T(v_i)}{dv_i} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial v_i}{\partial k_j} \\ \frac{\partial \lambda_i}{\partial k_j} \end{bmatrix} = \begin{bmatrix} \psi_3 v_i \\ 0 \end{bmatrix}, \tag{6.12}$$

where

$$\begin{aligned} \psi_1 &= \lambda_i I - A - Be^{-\lambda_i h_1} - Ce^{-\lambda_i h_2} - De^{-\lambda_i h_a} - Ee^{-\lambda_i h_s}, \\ \psi_2 &= I + h_1 Be^{-\lambda_i h_1} + h_2 Ce^{-\lambda_i h_2} + h_a De^{-\lambda_i h_a} + h_s Ee^{-\lambda_i h_s}, \\ \psi_3 &= \partial \left(-A - Be^{-\lambda_i h_1} - Ce^{-\lambda_i h_2} - De^{-\lambda_i h_a} - Ee^{-\lambda_i h_s} \right) / \partial k_j. \end{aligned}$$

Finally, using (6.12), one can compute $\frac{\partial \lambda_i}{\partial k_j}$ and thereby obtain S.
Before presenting the observations for different controllers, we first present a case to demonstrate the stabilizing ability of the CPPT. For implementation of the Algorithm 6.1, we set 1) Minimum real part of eigenvalue as 20, 2) Maximum number of eigenvalues to 10, 3) Maximum number of newton iteration to 12 in BIFTOOL. This parameter set are used for all the results presented in this chapter. Consider stabilization of system (6.1) with a first order Type II controller for the choice $h_a = h_s = h_1 = 0.50$. Variations of real parts of some rightmost eigenvalues is shown in Fig. 6.2. Initially, one pole is in $C_{\bar{r}}^{+}$ for a initial choice of $K = [1, \ -0.7, \ -2]$. We set $\bar{r} = -0.2$ and correspondingly $m = 1$. As the controller parameters update, the 2^{nd} rightmost complex pole pair splits into two real ones at the 28^{th} iteration. At 41^{st} iteration m is updated to 2 and the system is stabilized at 66 iteration

with a control gain $K = [1.1779, \quad -0.8958, \quad -2.0736]$.

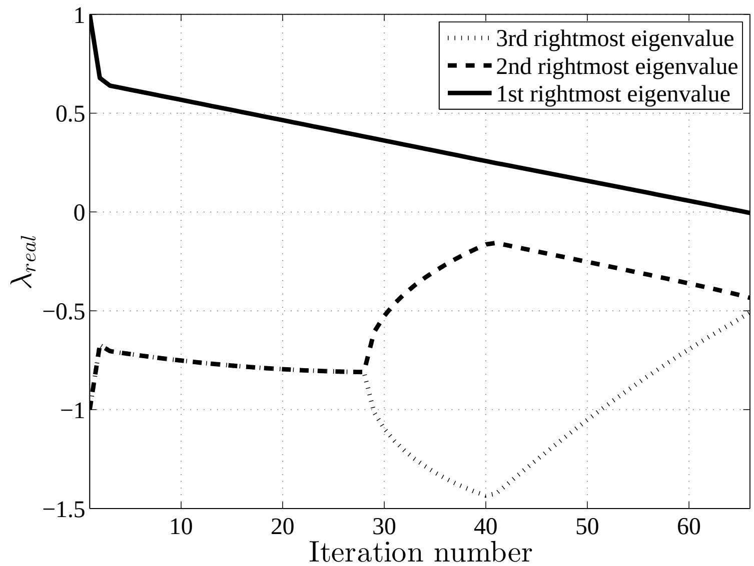

Figure 6.2: Variations of real parts of rightmost poles

6.4 Stabilization of scalar systems with input-output delay

This section presents stabilizing results obtained for the three controller types using CPPT for system (6.1) with $a = 1$. For this, as discussed, static state-feedback controllers can attain at most $h_{total} = 1$, whereas observer based controllers of [108] can attain $h_{total} = 2$. The approach is rigorous but effective to study the problem in hand. First order controller dynamics has been used for the study. Moreover, the results are obtained in the delay parameter plane to examine their ability to improve tolerable delay values.

6.4.1 Simple dynamic controller (Type I)

The variation of maximum tolerable h_s with respect to h_a for Type I controller is shown in Fig. 6.3. It is found that the maximum h_{total} is less than 1, i.e., the dynamic controller of the form (6.4) can stabilize the system upto the delay limit which is also attainable using static state-feedback controllers.

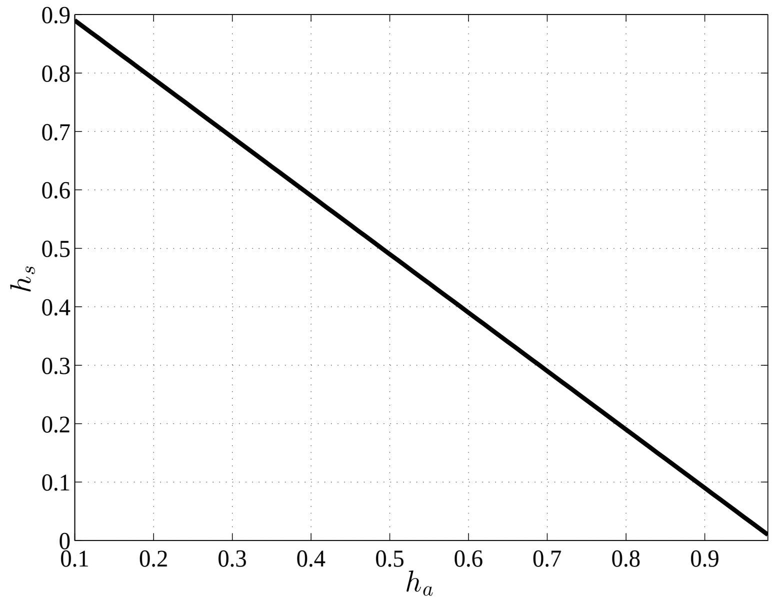

Figure 6.3: Variation of maximum h_s with respect to h_a using Type I controller

6.4.2 Dynamic controller with a state delay (Type II)

To study the behavior of a first order dynamic controller with a state delay of type-II, the structure of the controller is considered as, $\dot{x}_c(t) = a_{c0}x_c(t) + a_{c1}x_c(t-h_1) + c_c x_p(t-h_s)$, $u_p(t) = x_c(t)$. Using this controller, the closed loop system can be represented as: $\dot{\xi}(t) = A\xi(t) + B\xi(t-h_1) + D\xi(t-h_a) + E\xi(t-h_s)$, where $\xi(t) = \begin{bmatrix} x_p(t) \\ x_c(t) \end{bmatrix}$, $A = \begin{bmatrix} 1 & 0 \\ 0 & a_{c0} \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & a_{c1} \end{bmatrix}$, $D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $E = \begin{bmatrix} 0 & 0 \\ c_c & 0 \end{bmatrix}$. To obtain the controller parameters, K is arranged as $K = [a_{c0}, a_{c1}, c_c]$. To discuss the influence of state delay in the controller dynamics, the following subcases are considered.

6.4.2.1 Variation of $h_a = h_s$ with respect to h_1

The behavior of tolerable $h_a = h_s$ with respect to variation in h_1 is shown in Fig. 6.4. With increase in h_1 from 0.04, tolerable h_a (h_s) also gradually increases from a initial value of less than 0.5 until $h_1 = 1.74$, the corresponding maximum value of h_a (h_s) is 1.17, i.e. $h_{total} = 2.34$. Note that, this h_{total} is larger than the one achievable using static state-feedback (and simple dynamic feedback) controller and even using observer based controller of ($h_{total} = 2$) [108]. However, the improvement ability gets stalled at $h_1 = 1.74$. This point may be referred to as a *stalling point*. At this point, the convergence of the system state and controller state are shown in Fig. 6.5 and Fig. 6.6 with controller parameters $a_{c0} = -0.3247$, $a_{c1} = -3.5925$ and $c_c = -3.9173$, initial condition $x_p(t) = 1$, $t \in [-1.17, 0]$, $x_c(0) = 0$, $t \in [-1.74, 0]$. Beyond this point, the tolerable h_a first decreases abruptly and then remains at a value of $h_s = h_a \leq 0.5$ for a while. It is also observed that the tolerable $h_a(h_s)$ falls even below 0.5 with $h_1 > 30$.

Hence, the use of dynamic controllers with a state delay h_1 is beneficial compared to static controller or simple dynamic controllers provided h_1 is chosen suitably (within a certain range). Moreover, the optimal value of h_1 for which maximum tolerable delay obtained is highly fragile since a slight increase from this value decreases the tolerable delay values considerably.

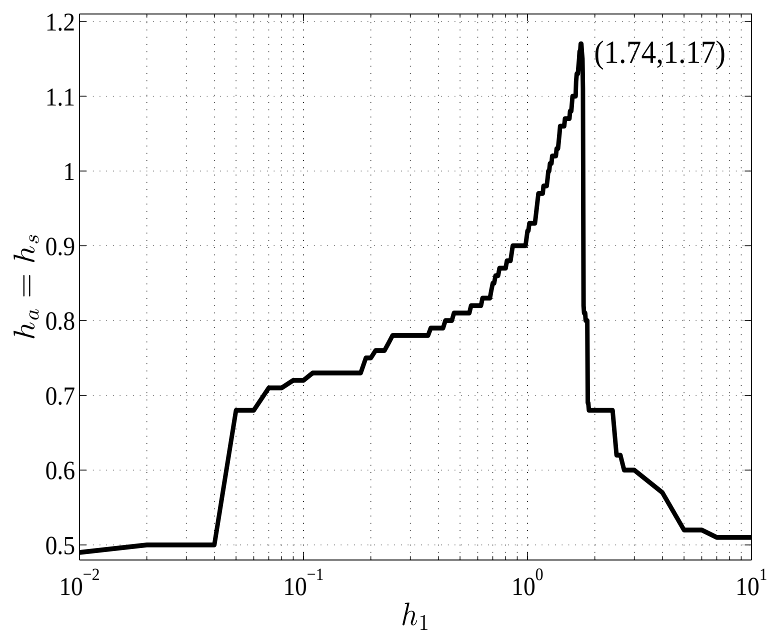

Figure 6.4: Variation of $h_a = h_s$ with respect to h_1 using Type II controller

6.4.2.2 Variation of h_a with respect to h_s for different values of h_1

Next, consider the variation of h_s with respect to h_a for different choices of h_1. Four different choices of $h_1 = 1.75, 1.74, 1.70$ and 1.50 (around the stalling point) are used for the study. The corresponding results are shown in Fig. 6.7. For a chosen h_1, h_{total} remains constant and $h_1 = 1.74$ is the optimal value for all combinations of h_a and h_s. It is also observed that a single controller works for a particular h_1. For example, the controller is as $a_{c0} = -0.3247, a_{c1} = -3.9173$ and $c_c = -3.5925$ for $h_1 = 1.74$. It may also be noted that unlike

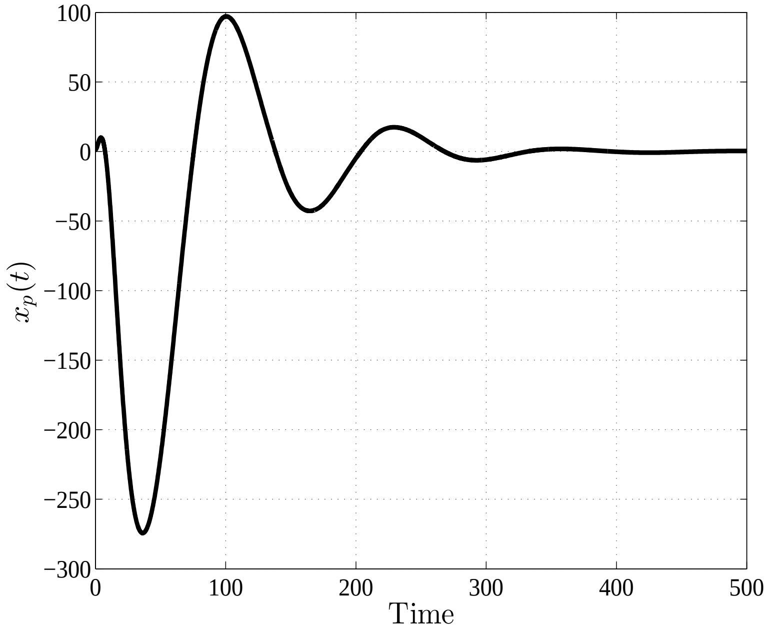

Figure 6.5: Variation of $x_p(t)$ with respect to time of scalar system using Type II controller

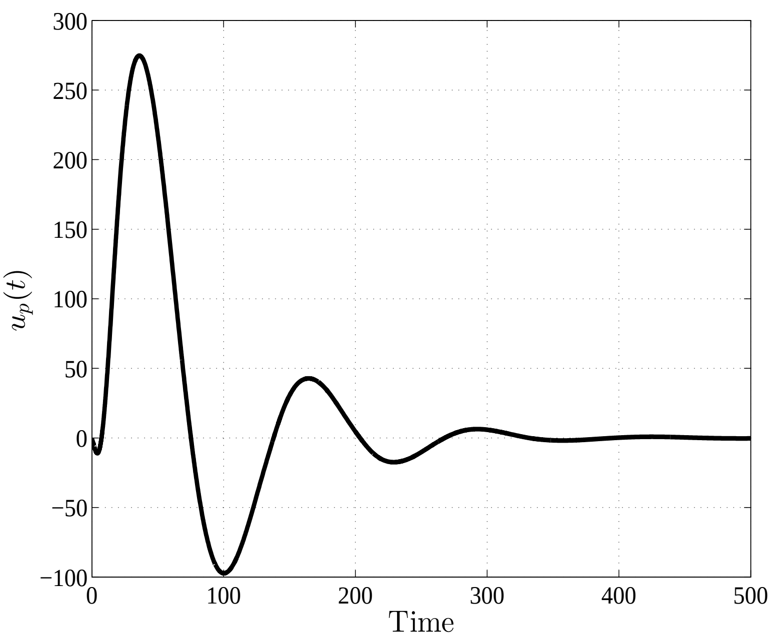

Figure 6.6: Variation of $u_p(t)$ (Type II controller) with respect to time for scalar system

observer based controller [108] for which $h_a \leq 1$ and $h_s \leq 1$, here the tolerable delay limit is only $h_{total} \leq 2.34$ irrespective of individual values of the delays. Therefore, the result in the previous subsection (Fig. 6.4) otherwise can be seen as the effect of h_1 on h_{total}.

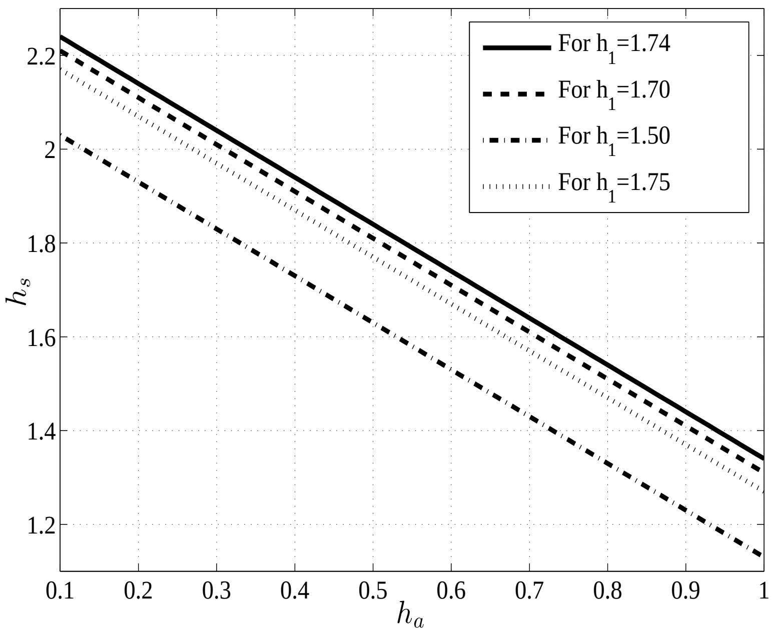

Figure 6.7: Variation of h_s with respect to h_a for different h_1 using Type II controller

6.4.2.3 Variation of h_a with respect to $h_1 = h_s$

The previous result does not answer the question whether choosing h_1 equal to h_a or h_s (provided they are known) is advantageous or not. To study this, consider the case that $h_1 = h_s$. For this, variation of tolerable h_a with respect to h_s is presented in Fig. 6.8. Initially, with increase in h_s until 0.21, h_a increases to 1.31, but after that it decreases gradually. At $h_s = 1.74$, h_a suddenly decreases to 0.17. Note that, this value is same as obtained in the previous sub-cases and corroborates that there exists an optimal h_1 for which maximum tolerable delay may be obtained. Same as the previous cases, the maximum h_{total} is 2.34 at $h_s = 1.74$. Therefore, the choice of optimal h_1 can be made independently. For a system of class (6.1) with $a = 1$, it is $h_1 = 1.74$.

6.4.3 Dynamic controller with two state delays (Type-III)

Next, to study the behavior of a first order dynamic controller with two state delays of type-III, the structure of the controller is considered as, $\dot{x}_c(t) = a_{c0}x_c(t) + a_{c1}x_c(t-h_1) + a_{c2}x_c(t-$

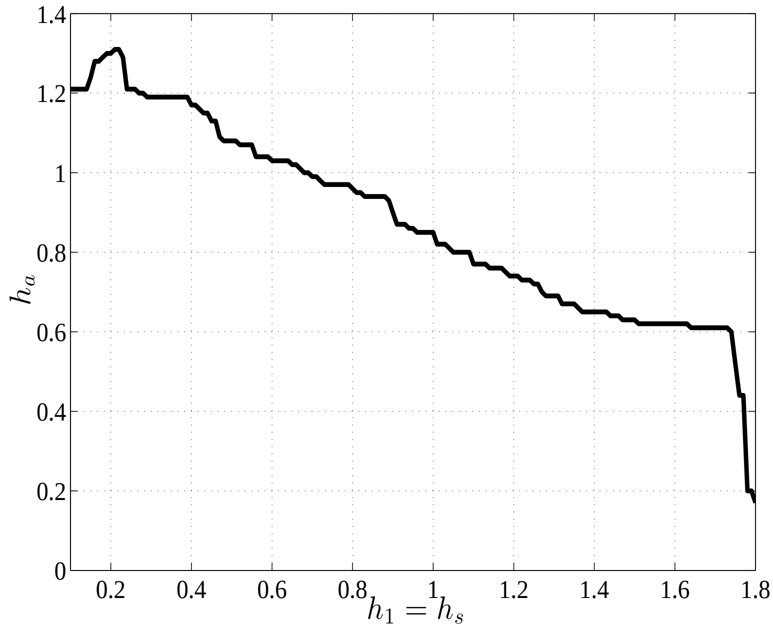

Figure 6.8: Variation of h_a with respect to $h_1 = h_s$ using Type II controller

$h_2) + c_c x_p(t - h_s)$, $u_p(t) = x_c(t)$. Using this dynamic controller, the closed loop system can be represented as: $\dot{\xi}(t) = A\xi(t) + B\xi(t - h_1) + C\xi(t - h_2) + D\xi(t - h_a) + E\xi(t - h_s)$, where $\xi(t) = \begin{bmatrix} x_p(t) \\ x_c(t) \end{bmatrix}$, $A = \begin{bmatrix} 1 & 0 \\ 0 & a_{c0} \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & a_{c1} \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ 0 & a_{c2} \end{bmatrix}$ $D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $E = \begin{bmatrix} 0 & 0 \\ c_c & 0 \end{bmatrix}$. To obtain the controller parameters, K is arranged as $K = [a_{c0}, a_{c1}, a_{c2}, c_c]$.

We consider here obtaining variation of tolerable $h_a = h_s$ with respect to h_2 for $h_1 = 1.74$ (the stalling point for Type II controller). This result is shown in Fig. 6.9. It is observed that for $h_2 \leq 1.89$, the tolerable $h_a = h_s$ is 1.17, which is same as that obtained using controller with a single delay. However, for $1.89 < h_2 \leq 2.54$, the h_{total} is larger compared to the single delay case. At this point, the convergence of the system state and controller state are shown in Fig. 6.10 and Fig. 6.11 with controller parameters $a_{c0} = -0.1704$, $a_{c1} = -3.3320$, $a_{c2} = -1.3767$ and $c_c = -4.8791$, initial condition $x_p(t) = 1$, $t \in [-1.35, 0]$, $x_c(0) = 0$, $t \in [-2.54, 0]$. Beyond this, the tolerable delay decreases significantly. The maximum tolerable delay is 2.70 for $h_2 = 2.54$.

6.5 Stabilization of a second order system

So far, we have studied the improvement abilities of dynamic state-feedback controller with state delays for scalar systems. To strengthen the observations further, a second order

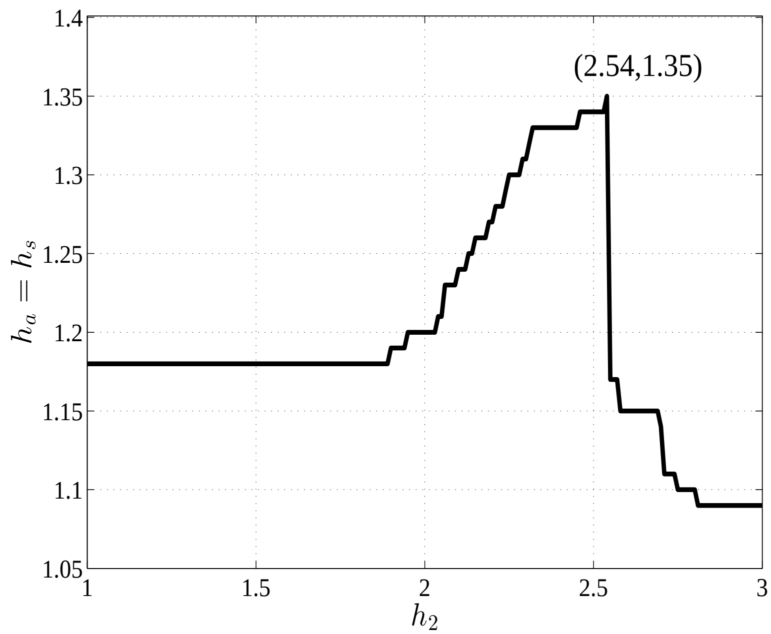

Figure 6.9: Variation of $h_a = h_s$ with respect to h_2 for $h_1 = 1.74$ using Type III controller

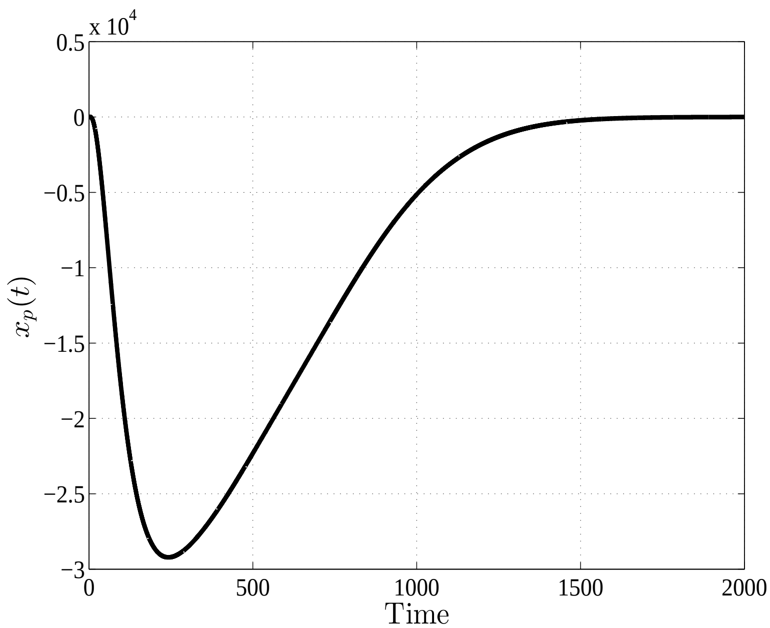

Figure 6.10: Variation of $x_p(t)$ with respect to time of scalar system using Type III controller

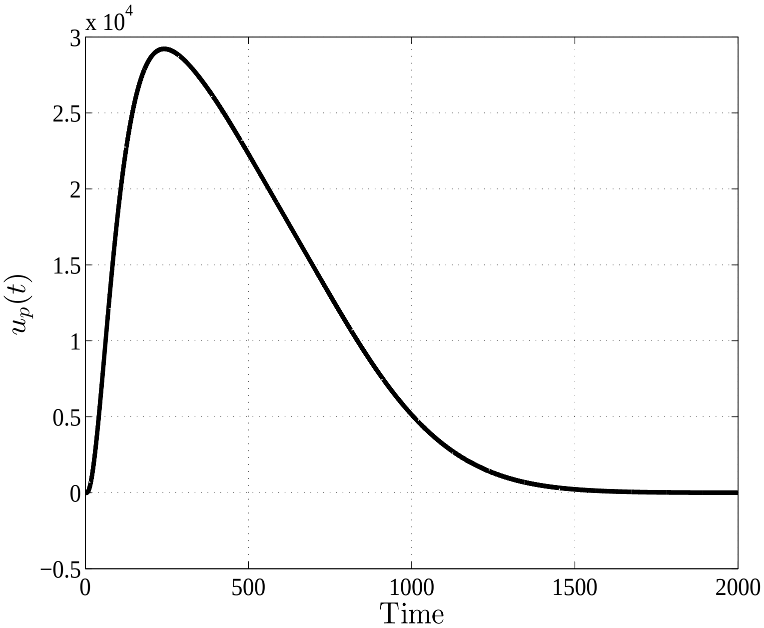

Figure 6.11: Variation of $u_p(t)$ (Type III controller) with respect to time for scalar system

example is now considered for stabilization with a dynamic controller of Type II. The plant matrices are given by

$$A_p = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix},\ B_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{6.13}$$

Let us consider a first order dynamic controller of type-II with a structure as follows.

$$\dot{x}_c = a_{c_0}x_c + a_{c_1}x_c(t-h_1) + c_{c_1}x_1(t-h_{total}) + c_{c_2}x_2(t-h_{total}), u = kx_c(t), \tag{6.14}$$

where h_{total} is the total tolerable delay, $h_{total} = h_s + h_a$. Using the dynamic controller (6.14), the closed loop system can be represented as:

$$\dot{\xi}(t) = A\xi(t) + C\xi(t-h_{total}) + D\xi(t-h_1), \tag{6.15}$$

where $\xi = \begin{bmatrix} x_1^T & x_2^T & x_c^T \end{bmatrix}^T$, x_1 and x_2 are two states of the plant and x_c is the state of the controller; $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & k \\ 0 & 0 & a_{c_0} \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_{c_1} & c_{c_2} & 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{c_1} \end{bmatrix}$.

We have arranged the controller parameters for designing the controller gain matrix

(K) as $K = \begin{bmatrix} k, & a_{c0}, & a_{c1}, & c_{c1}, & c_{c2} \end{bmatrix}$. Note that, the open-loop plant (6.13) has one real unstable eigenvalue at 0.6180. Following Theorem 7 of [109], (6.13) is stabilizable using any LTI controller for $h_{total} = 2/0.6180 = 3.2362$. Now, we study the variation of h_{total} with respect to h_1 using (6.14). The obtained maximum h_{total} is shown in Fig. 6.12. It can be seen that the tolerable delay (h_{total}) increases with respect to h_1 until $h_1 = 2.7$ (stalling point), but then decreases abruptly. At the stalling point, the total tolerable delay is 4.7, which is higher than that achievable using simple LTI controllers ($h_{total} = 3.2362$) [109]. At this point, the convergence of the system states and control input are shown in Fig. 6.13 and Fig. 6.14 with controller parameters $k = 0.85766$, $a_{c_0} = 0.347076$, $a_{c_1} = -1.852729$, $c_{c_1} = -1.75554$ and $c_{c_2} = -1.829$. Therefore, this example also substantiates the observations made previously that dynamic controllers with state delays has tolerable delay margin improvement ability.

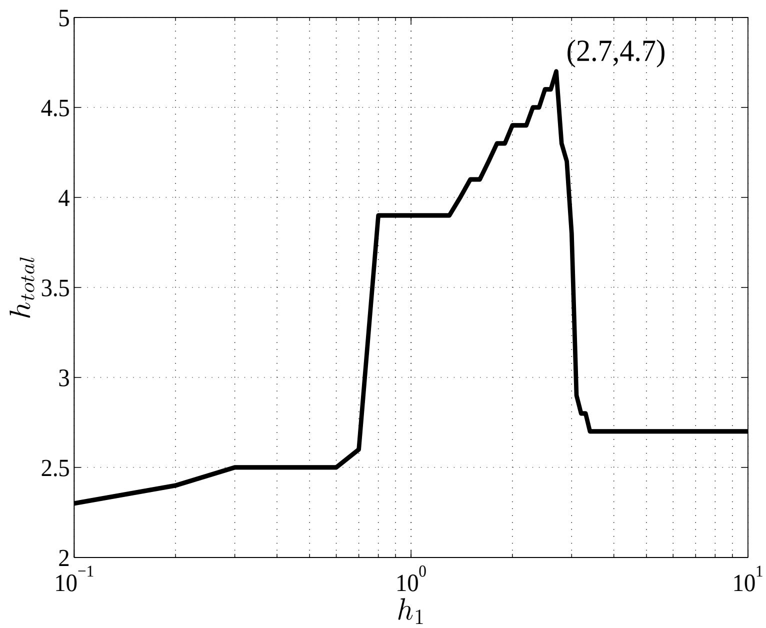

Figure 6.12: Variation of h_{total} with respect to h_1 for system 6.13 using a controller 6.14

6.6 Chapter summary

Stabilization improving ability of dynamic controllers with state delay is studied in this chapter. Using CPPT technique, stabilizing results are obtained in the delay-parameter

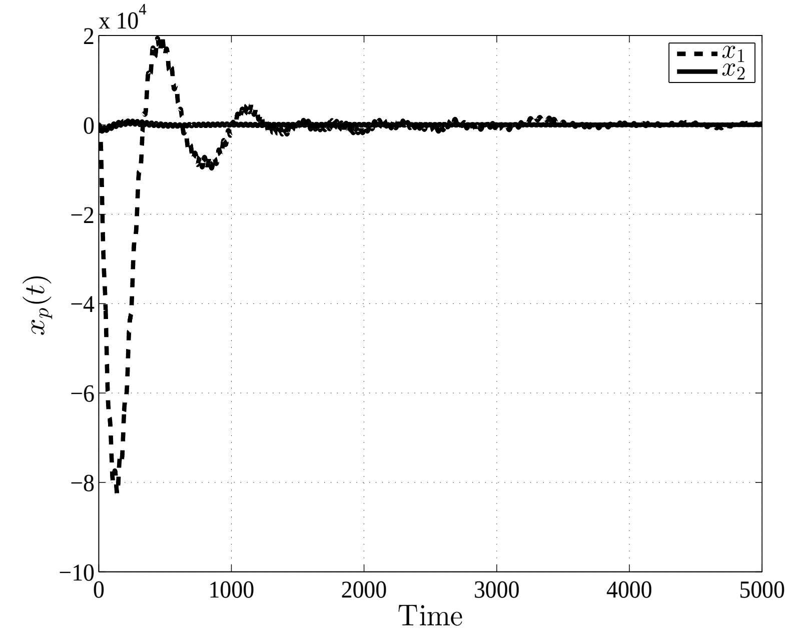

Figure 6.13: Variation of $x_p(t)$ with respect to time of second order system using Type II controller 6.14

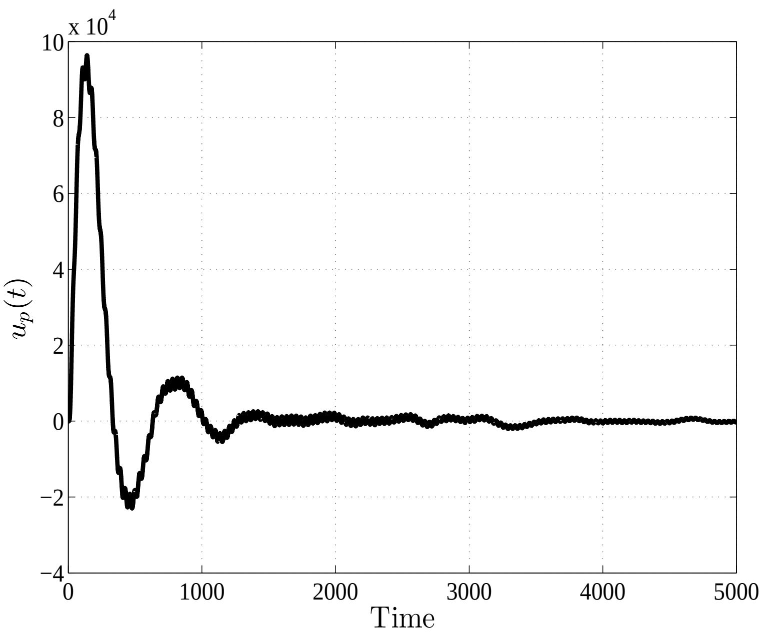

Figure 6.14: Variation of $u_p(t)$ (Type II controller 6.14) with respect to time for second order system

plane for two (one scalar and one second order) systems with input-output delays. The salient observations made from the studies are:

1. Use of dynamic state feedback controller with state delays may improve tolerable delay margin compared to static or simple dynamic feedback controllers.

2. Use of multiple delays may improve the tolerable delay margin further compared to using a single one.

CHAPTER 7

Conclusions and suggestions for future work

The stability analysis and stabilization of systems with delays are investigated in this thesis. New approaches have been proposed to obtain improved results on stability analysis and stabilization of such systems. The following section emphasizes chapterwise main contributions of the present work.

7.1 Thesis contributions

This thesis deals with some problems on stability analysis and stabilization of time-delay systems. For stability analysis of systems with constant delay, a delay-decomposition approach is proposed using simple LK functional which is further used for static state feedback stabilization of both for systems with state- and input-delay. Finally, the stabilizing ability of artificial delays incorporated in dynamic state feedback controller has been investigated.

A brief description about the contributions in each chapter are presented in the following.

- A delay-decomposition technique is proposed for systems with single, constant but uncertain delay in **Chapter 2** by considering a simple LK functional to derive a simple LMI condition. The dimension of the derived criterion is independent of the number of decomposition. So that the computational complexity of the criterion does not increase with respect to the number of decomposition. It has been shown through numerical examples that the present approach is superior over the existing ones. Finally, effectiveness of the proposed approach has been demonstrated for uncertain systems through numerical examples.

- The problem of stability analysis of systems with two constant delays has been addressed in **Chapter 3**. The extraction of overlapping feature of the delays to derive less conservative result for such systems, that appears to be not addressed so far in literature, has been considered in this chapter. By exploiting this feature, a less conservative criterion is obtained as compared to the existing results. Then the approach is extended for robust stability analysis. For both cases, the effectiveness of the proposed approach for systems with two delays has also been demonstrated through numerical examples.

- The **Chapter 4** considers the state feedback stabilization of systems with single delay using the decomposition approach proposed in Chapter 2. As the approach involves a simple LK functional, the derived criterion becomes finite-dimensional and it is independent of the number of decomposition. For easy implementable capability of the approach, it has also been used for control design of uncertain system. Though the approach leads to a sufficient stabilization criteria, but they are less conservative compared to the existing finite-dimensional approaches. To validate the effectiveness of the derived criteria, numerical examples are presented for comparative analysis.

- The control design for input-delay systems are challenging issues. To deal with these issues, the **Chapter 5** includes LMI based control design approaches for such systems. A simple linearization technique are applied to derive less conservative static state feedback stabilization criterion. To further improve the stabilizing ability, the dimension of the designed controller is increased by introducing a state feedback PI-type controller. Finally, the decomposition approach proposed in Chapter 2 is implemented to get a less conservative criterion. All the proposed criteria in this chapter is tested on numerical examples to check the effectiveness of the approaches.

- The stabilizing ability of artificial delay is investigated in **Chapter 5**. Using continuous pole placement technique technique, stabilizing results are obtained in the delay-parameter plane for systems with input-output delay. The improvement of tolerable delay bound using dynamic controllers with artificial state delay as compared to static or simple dynamic feedback controllers is studied through a scalar case. And a study has been made that the stabilizing ability of the dynamic controller with multiple state delays may improve the tolerable delay margin.

7.2 Suggestions for future work

The thesis opens up some scope for future work which has been given below.

- The delay-decomposition technique proposed for systems with constant delay may be investigated for systems with time-varying delay. The challenge that one may face on this problem is in developing the theory based on multiple LK functional. This remains an open problem.

- To design a controller for a system with single and constant delay, the system is considered to be fully controllable and the states are measurable for feedback. Therefore, the proposed delay-decomposition approach is used in Chapter 4 to design a state-feedback delay-controller for the system with single and constant delay. For a output feedback case, the output vector ($y(t)$) will be in the form of $y(t) = Cx(t)$. Due to the presence of this C matrix, the non-linear matrix inequality for stabilization is difficult to handle by existing linearization of matrix inequalities techniques. Therefore, the use of proposed delay-decomposition technique for partial or output feedback controller design remains an open problem for further investigation.

- The proposed overlapping approach for stability analysis of systems with two delays may be extended for deriving less conservative stabilization criterion for systems with arbitrary number of delays. However, for arbitrary number of delays, the overlapping information becomes complicated. How to reduce this complexity is an open issue.

- Besides the investigation made on stabilizing ability of the delayed state dynamic controller, the work opens up a new set of problems involving the proposed dynamic controllers, for example, how to design such controllers using Lyapunov framework, how is the effect of such controllers on plant uncertainties, to mention a few.

Appendix

Linear Matrix Inequality [6, 141]

All the stability and stabilization criterion discussed in the preceding section as well as many other problems in systems and control can be formulated as optimization problems involving constraints that can be expressed as LMIs having the following form:

$$F(x) = F_0 + \sum_{i=1}^{p} x_i F_i < 0, \tag{7.1}$$

where $x \in \Re^p$ is the variable vector and x_i being the i^{th} element of it, $F_i = F_i^T \in \Re^{q \times q}$, $i = 0, 1, \ldots, p$ are constant known matrices where q is a positive integer. Clearly, a set of LMIs can easily be expressed as a single LMI.

The important property of (7.1) is that this defines a convex constraint on the variable x. Now, if the objective function of an optimization problem is convex and the constraints are in LMI form then the whole problem can be cast as a convex optimization problem in LMI framework. Note that, convex optimization problems are attractive mainly for two reasons: (a) local minima is the global minima and it is unique if it exists and (b) computationally attractive due to available efficient algorithms for solving these.

In fact problems associated with LMI can be classified into three categories:

1. *Feasibility problem:*

   Finding if there exists a solution of an LMI ($F(x) < 0$).

2. *Optimization problem:*

   Minimizing a convex objective $f(x)$ subject to an LMI constraint ($F(x) < 0$).

3. *Generalized eigenvalue problem:*

   Minimizing λ subject to $G(x) - \lambda F(x) < 0$, $F(x) > 0$ and $H(x) < 0$.

Often, a class of nonlinear matrix inequalities are confronted in systems and control theory which can be reformulated as LMIs using *Schur Complement* formula [6]. It states that for matrices $Z_1 = Z_1^T$, $Z_2 = Z_2^T$ and L,

$$\left.\begin{array}{l} Z_2 < 0 \quad \text{and} \quad Z_1 - L Z_2^{-1} L^T < 0, \\ \text{is equivalent to} \begin{bmatrix} Z_1 & L \\ L^T & Z_2 \end{bmatrix} < 0, \end{array}\right\}. \tag{7.2}$$

The LMI Control Toolbox of MATLAB® [35]

The LMI control toolbox provides an LMI Lab to specify and solve user defined LMIs. In this thesis, this LMI Lab has been used for solving LMIs. Some commands of this LMI Lab that are used for producing the numerical results are presented in the following.
SETLMIS : This initializes the LMI system description.
GETLMIS : It is used when all the LMIs are described and returns the internal description of the defined LMI.
LMIVAR : It is used to declare the LMI variables.
LMITERM: The LMI terms are specified with this command.
FEASP : This is an LMI solver which is used to solve LMI feasibility problems.
MINCX : This LMI solver is used to solve an LMI optimization problem.
GEVP : It is used for solving generalized eigenvalue problem.
A graphical user interface LMIEDIT also exists to define LMIs.

The *FMINSEARCH* command of the Optimization Toolbox of MATLAB® [18] has also been used to tune certain parameters that are associated with the derived LMI criteria.

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